Pairing in many-fermion systems: an exact-renormalisation-group treatment

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Birse, Krippa, McGovern and Walet, hep-ph/0406249

ERG method:

Berges, Tetradiis and Wetterich, hep-ph/0005122
Delamotte, Mouhanna and Tissier, cond-mat/0309101
Background

Ideas of effective field theory and renormalisation group now well-developed for few-nucleon systems
- rely on separation of scales
- RG can be used to derive power counting
  → classify terms as perturbations around fixed point
- consistent extension of old ideas (effective-range expansion)

Many unsuccessful attempts to extend to nuclear matter
- problem: no separation of scales
- only consistent EFT so far: weakly repulsive Fermi gas
  [Hammer and Furnstahl, nucl-th/0004043]
  (reproduces old results of Bishop and others)
Other EFT’s for interacting Fermi systems exist:
– Landau Fermi liquid [Shankar], Ginsburg-Landau theory
– but parameters have no simple connection to underlying forces (like ChPT and QCD)

Look for some more heuristic approach
– based on field theory
– can be matched onto EFT’s for few-nucleon systems
– input from two-body (and three-body) systems in vacuum

Try “exact” renormalisation group
– based on Wilsonian RG approach to field theories
– successfully applied to various systems in particle physics and condensed-matter physics

[version due to Wetterich, Phys Lett B301 (1993) 90]
Outline

- ERG for the effective action
- Effective action for fermions with attraction
- Choice of regulator functions
- Evolution equations: general structure
- Driving terms in evolution equations
- Initial conditions
- Results
ERG for the effective action

For single real scalar field $\phi$ start from

$$e^{iW[J]} = \int D\phi e^{i(S[\phi] + J\cdot\phi - \frac{1}{2}\phi\cdot R\cdot \phi)}$$

$R(q, k)$: regulator function for the ERG
- suppresses contributions of modes with low momenta, $q \lesssim k$
- only modes with $q \gtrsim k$ integrated out
- $W[J]$ evolves with regulator ("cut-off") scale $k$
- becomes full generating function as $k \to 0$

Legendre transform $\rightarrow$ effective action $\Gamma[\phi_c]$ (generator for 1PI diagrams)
where expectation value of the field is

$$\frac{\delta W}{\delta J} \equiv \phi_c$$

[See talks by: Furnstahl, Litim, Polonyi]
Effective action

[convention as in: Weinberg, Quantum Theory of Fields II]

\[ \Gamma[\phi_c] = W[J] - J \cdot \phi_c + \frac{1}{2} \phi_c \cdot R \cdot \phi_c \]

\( W \) evolves with scale \( k \) according to

\[ \partial_k W = -\frac{1}{2} \phi_c \cdot \partial_k R \cdot \phi_c + \frac{i}{2} \text{Tr} \left[ (\partial_k R) \frac{\delta \phi_c}{\delta J} \right] \]

Evolution of \( \Gamma[\phi_c] \)

- \( J \) also runs if \( \phi_c \) is held constant
- \( \phi_c \cdot \partial_k R \cdot \phi_c \) terms cancel

\[ \partial_k \Gamma = \frac{i}{2} \text{Tr} \left[ (\partial_k R) \frac{\delta \phi_c}{\delta J} \right] \]
From definition of $\Gamma$

$$\frac{\delta J}{\delta \phi_c} = - (\Gamma^{(2)} - R)$$

where

$$\Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta \phi_c \delta \phi_c}$$

→ evolution equation for $\Gamma$ in form of a one-loop integral

$$\partial_k \Gamma = -i \frac{1}{2} \text{Tr} \left[ (\partial_k R) (\Gamma^{(2)} - R)^{-1} \right]$$

$(\Gamma^{(2)} - R)^{-1}$: propagator of boson in background field $\phi_c$

One-loop structure: like RG for few-body systems

→ can match ERG in matter onto interactions in vacuum

[Polchinski’s version of ERG: different structure

– see for example: Morris, hep-th/9802039]
For a system of fermions as well as bosons

\[ \partial_k \Gamma = +\frac{i}{2} \text{Tr} \left[ (\partial_k R_F) \left( (\Gamma^{(2)} - R)^{-1} \right)_{FF} \right] \]
\[ -\frac{i}{2} \text{Tr} \left[ (\partial_k R_B) \left( (\Gamma^{(2)} - R)^{-1} \right)_{BB} \right] \]

System with charged condensates (pairing → particle-hole mixing)
– write propagator for complex field as \( 2 \times 2 \) matrix (Nambu-Gor’kov)
→ factors of \( \frac{1}{2} \) still present
Regulator function and ansatz for $\Gamma$

Regulator $R(q, k)$: IR cut-off on effective action $\Gamma$
– should suppress contributions of modes with $q \lesssim k$
– should give back full effective action as $k \to 0$
\[ \to R(q, k) \text{ should provide large mass/energy gap for modes with } q \lesssim k \]
and should vanish for $q \gg k$ and $k \to 0$

Derivative $\partial_k R(q, k)$ in ERG equation
– peaks for $q \sim k$
– tends to zero for $q \gg k$
\[ \to \text{acts as UV cut-off on loop integrals} \]

ERG: complicated differential equation for functional $\Gamma$
– need to choose an ansatz for effective action
– make an expansion in local terms (as in rigorous EFT’s)
– use physics to guide choice
Effective action for fermions with attraction

Attractive forces between fermions → pairing [Furnstahl, Hands]
- weak attraction: Cooper pairs (BCS state) $\mu \simeq \epsilon_F$
- strong attraction: Bose-Einstein condensation (BEC) $\mu < 0$

Single species of nonrelativistic fermion: $\psi$ (as in neutron matter)
Boson field describing correlated fermion pairs: $\phi$
Finite density: chemical potential $\mu$

Ansatz for $\Gamma$:

$$
\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger, \mu, k] = \int d^4 x \left[ \phi^\dagger(x) \left( Z_\phi (i\partial_t + 2\mu) + \frac{Z_m}{2m} \nabla^2 \right) \phi(x) - U(\phi, \phi^\dagger) \\
+ \psi^\dagger \left( Z_\psi (i\partial_t + \mu) + \frac{Z_M}{2M} \nabla^2 \right) \psi \\
- Z_g g \left( \frac{i}{2} \psi^T \sigma_2 \phi \phi^\dagger - \frac{i}{2} \psi^\dagger \sigma_2 \psi \phi^T \phi \right) \right]
$$
Potential:
Bosons carry twice charge of a fermion
– couple to chemical potential \( \mu \) via quadratic term
– absorb into potential

\[
\bar{U} = U - 2\mu Z \phi \phi \phi^\dagger\phi
\]

Expand potential about minimum \( \phi^\dagger \phi = \rho_0 \) to quadratic order:

\[
\bar{U} = u_0 + u_1 (\phi^\dagger \phi - \rho_0) + \frac{1}{2} u_2 (\phi^\dagger \phi - \rho_0)^2
\]

(one redundant parameter: \( \rho_0 \) or \( u_1 \))

In symmetric phase: \( \rho_0 = 0 \)
In condensed phase: \( u_n \) defined at minimum \( \rightarrow u_1 = 0 \)
In condensed phase with uniform background \( \phi \) field:

Particles and holes mix \((\psi \text{ and } \psi^{\dagger} \text{ coupled})\)
\( \rightarrow \) fermion spectrum with energy gap \( \Delta = g|\phi|/Z_{\phi} \)

\[
E_{F}(q) = \pm \frac{1}{Z_{\psi}} \sqrt{\left(\frac{ZM}{2M}(q^{2} - p_{F}^{2})\right)^{2} + g^{2}\phi^{\dagger}\phi}
\]

Bosons become gapless Goldstone modes
– spectrum \((\phi \text{ and } \phi^{\dagger} \text{ also coupled})\)

\[
E_{B}(q) = \pm \frac{1}{Z_{\phi}} \sqrt{\frac{Zm}{2m}q^{2} \left(\frac{Zm}{2m}q^{2} + 2u_{2}\phi^{\dagger}\phi\right)}
\]

\( \rightarrow \) superfluid state: BCS or BEC
Γ depends on cut-off scale $k$ through running:

- coefficients in potential, $u_0, u_1$ (or $\rho_0$), $u_2$
- wave-function renormalisation factors, $Z_\phi, Z_\psi$
- mass renormalisations, $Z_M, Z_m$
- coupling constant renormalisation, $Z_g$

To study crossover from BCS pairing to BEC
- need to work at fixed density
  (otherwise can’t get to negative $\mu$ for BEC)
→ must allow $\mu$ to run with $k$

ERG becomes a set of coupled first-order ODE’s
**Bare theory:** at starting scale $k = K$

- two-body interaction between fermions only

$$L_{\text{int}} = -\frac{1}{4} C_0 \left( \psi \sigma_2 \psi^\dagger \sigma_2 \right) \left( \psi^T \sigma_2 \psi \right)$$

- bosons just auxiliary fields (Hubbard-Stratonovich)

$$C_0(k) = -\frac{g(k)^2}{u_1(k)}$$

and $Z_{\phi,m}(k) \ll 1$, $u_2(k) \ll |C_0|$

(separation of $C_0$ arbitrary $\rightarrow$ results independent of $g(k)$)

Fermions not dressed at $k = K \rightarrow Z_\psi(k) = Z_M(k) = Z_g(k) = 1$

Here (first study):

- allow only potential $(u_n, \rho_0)$ and $Z_\phi$ to run independently

- freeze $Z_\psi = Z_M = Z_g = 1$ and set $Z_m = Z_\phi$ or 1
Choice of regulator functions

Nonrelativistic systems
– carry out loop integrals over energy exactly
– regulate only integrals over three-momentum

Bosonic regulator:

\[ R_B(q, k) = \frac{k^2}{2m} f(q/k) \]

where \( f(x) \to 1 \) as \( x \to 0 \) and \( f(x) \to 0 \) as \( x \to \infty \) (and \( q = |q| \))

Take \( R_B(q, k) \propto k^2 \) for \( q \lesssim k \)
→ large-\( k \) behaviours of integrals reflect UV divergences

Here: use smoothed step function

\[ f(q/k) = \frac{1}{2 \text{erf}(1/\sigma)} \left[ \text{erf} \left( \frac{q + k}{k\sigma} \right) + \text{erf} \left( \frac{q - k}{k\sigma} \right) \right] \]

\( \sigma \): parameter controlling sharpness
**Fermionic regulator:**

- should be positive for particle states \( q^2/2M > \mu \)
- and negative for hole states \( q^2/2M < \mu \)

(can’t just add artificial gap term – regulator must work in vacuum)

Here: use

\[
R_F(q, p_F, k) = \text{sgn}(q - p_F) \frac{k^2}{2M} f\left(\frac{q - p_F}{k}\right)
\]

\( p_\mu = \sqrt{2M\mu} \): Fermi momentum corresponding to running \( \mu \)

\( p_F = (3\pi^2 n)^{1/3} \): related to density \( n \)

**Symmetric phase:** \( p_F = p_\mu \) (until \( Z_{\psi,M} \) run)

**Condensed phase:** “Fermi surface” no longer at \( p_F \)

(not even well-defined for large gaps)

but gap in fermion spectrum \( \rightarrow \) regulator no longer crucial
Evolution equations: general structure

At present level of truncation (running \( u_n \), \( \rho_0 \) and \( Z_\phi \) only):

– all equations obtained from effective potential for uniform \( \phi \) field

– evolves according to

\[
\partial_k \bar{U} = -\frac{1}{\mathcal{V}_4} \partial_k \Gamma
\]

\( \mathcal{V}_4 \): volume of spacetime

Write potential in terms of \( \rho = \phi^\dagger \phi \): \( \bar{U}(\rho, \mu, k) \)

– coefficients

\[
u_n = \frac{\partial^n \bar{U}}{\partial \rho^n} \bigg|_{\rho=\rho_0}
\]

– density and wave-function renormalisation

\[
n = -\frac{\partial \bar{U}}{\partial \mu} \bigg|_{\rho=\rho_0}
\]

\[
Z_\phi = -\frac{1}{2} \frac{\partial^2 \bar{U}}{\partial \rho \partial \mu} \bigg|_{\rho=\rho_0}
\]
All quantities defined at running minimum $\rho = \rho_0(k)$

$\rightarrow$ extra implicit dependence on $k$ in condensed phase

$\rightarrow$ evolution of $u_n$ at constant $\mu$:

$$\partial_k u_n - u_{n+1} \partial_k \rho_0 = \left. \frac{\partial^n}{\partial \rho^n} \left( \partial_k \bar{U} \right) \right|_{\rho=\rho_0}$$

$\rightarrow$ couples $u_2$ to $u_3$: beyond current level of truncation

Could simply set $u_3 = 0$, but can do better:

$\rightarrow$ take $u_3(k)$ from evolution with fermion loops only

(can be solved analytically)

$\rightarrow$ approximation becomes exact if boson loops negligible
Evolution at constant density:

Running $\mu(k) \rightarrow$ further implicit dependence on $k$

Define total derivative

$$d_k = \partial_k + (d_k\mu) \frac{\partial}{\partial \mu}$$

and apply to $\partial \bar{U}/\partial \mu \rightarrow$ evolution equation for density

$$d_k n - 2Z\phi d_k \rho_0 + \chi d_k \mu = -\frac{\partial}{\partial \mu} \left( \partial_k \bar{U} \right) \bigg|_{\rho = \rho_0}$$

where fermion-number susceptibility is

$$\chi = \frac{\partial^2 \bar{U}}{\partial \mu^2} \bigg|_{\rho = \rho_0}$$
Keep $n$ constant $\rightarrow$ coupled equation for $\rho_0$ and $\mu$

$$-2Z_\phi d_k \rho_0 + \chi d_k \mu = -\frac{\partial}{\partial \mu} \left( \frac{\partial k \bar{U}}{\partial \rho} \right) \bigg|_{\rho=\rho_0}$$

**Symmetric phase:** driving term on RHS vanishes and $\rho_0 = 0$  
$\rightarrow$ evolution at constant $n$ same as at constant $\mu$  
$\rightarrow$ much simpler set of evolution equations:

$$\partial_k u_1 = \frac{\partial}{\partial \rho} \left( \frac{\partial k \bar{U}}{\partial \rho} \right) \bigg|_{\rho=0}$$

$$\partial_k u_2 = \frac{\partial^2}{\partial \rho^2} \left( \frac{\partial k \bar{U}}{\partial \rho} \right) \bigg|_{\rho=0}$$

$$\partial_k Z_\phi = -\frac{1}{2} \frac{\partial^2}{\partial \mu \partial \rho} \left( \frac{\partial k \bar{U}}{\partial \rho} \right) \bigg|_{\rho=0}$$
Condensed phase: set of equations is

\[-u_2 d_k \rho_0 + 2 Z_\phi d_k \mu = \frac{\partial}{\partial \rho} \left( \frac{\partial_k \bar{U}}{\rho = \rho_0} \right)\]

\[d_k u_2 - u_3 d_k \rho_0 + 2 Z'_\phi d_k \mu = \frac{\partial^2}{\partial \rho^2} \left( \frac{\partial_k \bar{U}}{\rho = \rho_0} \right)\]

\[d_k Z_\phi - Z'_\phi d_k \rho_0 + \frac{1}{2} \chi' d_k \mu = -\frac{1}{2} \frac{\partial^2}{\partial \mu \partial \rho} \left( \frac{\partial_k \bar{U}}{\rho = \rho_0} \right)\]

Here

\[Z'_\phi = -\frac{1}{2} \left. \frac{\partial^3 \bar{U}}{\partial \rho^2 \partial \mu} \right|_{\rho = \rho_0}\]

\[\chi' = \left. \frac{\partial^3 \bar{U}}{\partial \mu^2 \partial \rho} \right|_{\rho = \rho_0}\]

– like \(u_3\) and \(\chi\): beyond current level of truncation
→ replace by expressions from fermion loops only
Driving terms in evolution equations

Right-hand sides of evolution equations
– all obtained from $\partial_k \bar{U}$: sum of fermion and boson one-loop integrals

Treat $\psi$ and $\psi^\dagger$ as independent fields (also $\phi$ and $\phi^\dagger$)
→ 2 × 2 matrix structure for Gor’kov propagators etc

In presence of uniform $\phi$ field
– inverse fermion propagator

\[
\Gamma_{FF}^{(2)} - R_F = \begin{pmatrix}
Z_\psi q_0 - E_{FR} + i\epsilon \text{sgn}(q - p_\mu) & ig\phi\sigma_2 \\
-ig\phi^\dagger \sigma_2 & Z_\psi q_0 + E_{FR} - i\epsilon \text{sgn}(q - p_\mu)
\end{pmatrix}
\]

where

\[
E_{FR}(q, p_F, k) = \frac{1}{2M}q^2 - \mu + R_F(q, p_F, k) \text{sgn}(q - p_\mu)
\]
Inverse boson propagator

\[
\Gamma^{(2)}_{BB} - R_B = \begin{pmatrix}
Z\phi q_0 - E_{BR} + i\epsilon & -u_2\phi\phi \\
-u_2\phi^\dagger\phi^\dagger & -Z\phi q_0 - E_{BR} + i\epsilon
\end{pmatrix}
\]

where

\[
E_{BR}(q, k) = \frac{Z_m}{2m} q^2 + u_1 + u_2(2\phi^\dagger\phi - \rho_0) + R_B(q, k)
\]

Evaluating the loop integrals gives (after some work)

\[
\partial_k \bar{U} = -\frac{1}{V_4} \partial_k \Gamma = -\int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{E_{FR}}{\sqrt{E_{FR}^2 + \Delta^2}} \text{sgn}(q - p_\mu) \partial_k R_F
\]

\[
+ \frac{1}{2Z_\phi} \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{E_{BR}}{\sqrt{E_{BR}^2 - V_B^2}} \partial_k R_B
\]

where \( \Delta^2 = g^2\phi^\dagger\phi \) and \( V_B = u_2\phi^\dagger\phi \)

Derivatives of \( \partial_k \bar{U} \) with respect to \( \rho = \phi^\dagger\phi \) and \( \mu \to \) driving terms
Initial conditions

Run evolution down to \( k = 0 \) starting from some large scale \( k = K \)

Initial conditions obtained by matching onto evolution in vacuum for \( k \geq K \)
– fermion loops only
– can be integrated analytically to get

\[
\frac{u_1(K)}{g^2} = -\frac{M}{4\pi a} + \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \frac{1}{E_{FR}(q, 0, 0)} - \frac{1}{E_{FR}(q, 0, K)} \right]
\]

since \( u_1(0) \) related to physical scattering amplitude at threshold by

\[
T_F = \frac{4\pi a}{M} = -\frac{g^2}{u_1(0)} \quad \alpha: \text{scattering length}
\]

Both integrals diverge linearly on their own
– usual linear divergence in EFT’s for two-body scattering
→ difference linear in \( K \)
   (chose \( R_F \propto k^2 \) for large \( k \rightarrow K \sim \text{cut-off scale} \))
Need to be careful in matter: regulator shifted by Fermi surface
– acts like cut-off at $K + p_F$ for large $K \gg p_F$
→ constant shift $\propto p_F$ in linearly divergent integral
→ define $u_1(K)$ for use in matter by

$$\frac{u_1(K)}{g^2} = -\frac{M}{4\pi a} + \frac{1}{2} \int \frac{d^3 \vec{q}}{(2\pi)^3} \left[ \frac{1}{E_{FR}(q,0,0)} - \frac{\text{sgn}(q - p_F)}{E_{FR}(q,p_F,K)} \right]$$

– like using regulator that interpolates smoothly between
$R_F(q,p_F,k)$ for $k \gg p_F$ and $R_F(q,0,k)$ for $k \lesssim p_F$
(Fermi sea in second term $\sim$ totally suppressed for $K \gg p_F$)

Other initial values: $u_2(K)$, $Z_\phi(K)$ determined similarly
(but $p_F$-dependence suppressed by powers of $p_F/K$)
Results

Technique can be applied to many systems
– for definiteness start with parameters relevant to neutron matter:
  \[ M = 4.76 \text{ fm}^{-1}, \ p_F = 1.37 \text{ fm}^{-1}, \ |a| \gg 1 \text{ fm} \]
– then explore wider range of values for \( p_F a \)

Results are independent of \( K \) for \( K \gtrsim 4p_F \)
  (provided we are careful to use shifted cut-off to define \( u_1(K) \))
Also independent of width parameter \( \sigma \) in regulator functions

Compare results with simpler approximation keeping fermion loops only
– mean-field approximation for bosons
→ analytic results
  [Marani, Pistolesi and Strinati, cond-mat/9703160
  Papenbrock and Bertsch, nucl-th/9811077
  Babaev, cond-mat/0010085]
Mean-field effective potential (at $k = 0$)

$$U^\text{MF}(\Delta, \mu) = \frac{k_\Delta^5}{2M\pi} \left[ \frac{1}{8ak_\Delta} - \frac{1}{15}(1 + x^2)^{\frac{3}{4}} \frac{P^1_3}{2} \left( - \frac{x}{\sqrt{1 + x^2}} \right) \right]$$

$P^m_l(y)$: associated Legendre function

$k_\Delta = \sqrt{2M\Delta}$, $x = \mu/\Delta$, in terms of gap $\Delta = g|\phi|$

Minimise with respect to $\Delta$ at constant density
→ nonlinear equations for $\Delta$, $\mu$

In limit of weak attraction, $p_Fa \to 0^-$, gap has exponential form

$$\Delta \simeq \frac{8}{e^2} \varepsilon_F \exp \left( - \frac{\pi}{2p_F|a|} \right)$$
Numerical solutions to the evolution equations for infinite $a_0$, starting from $K = 16 \text{ fm}^{-1}$ (all in appropriate powers of fm$^{-1}$)

—: full solution  
—: fermion loops only

Transition to condensed phase ($u_1 = 0$) at $k_{\text{crit}} \simeq 1.2 \text{ fm}^{-1}$

Contributions of boson loops small (negligible in symmetric phase)
Crossover from BCS to BEC

Gap $\Delta$ and chemical potential $\mu$
- at physical point ($k = 0$)
- over wide range of densities or couplings, $(p_F a)^{-1}$

- fermions only (analytical)
- fermions only (numerical)
- bosonic loops with $Z_\phi = 1$
- full results

BCS: positive $\mu \sim p_F^2/2M$
(large negative $(p_F a)^{-1}$)

BEC: large negative $\mu$
(large positive $(p_F a)^{-1}$)
Closer look at gap

Fractional deviation of gap from analytic mean-field result

- : fermions only (numerical)
• : bosonic loops, $Z_\phi = 1$
×: full results
**Comments**

In region of strong attraction
- contributions of boson loops to gap $\Delta$ very small
- $\sim 1\%$ enhancement of gap in large-$pFa$ region
- effects on other quantities larger ($\sim 10\%$ in $u_2$)
- tend to cancel in $\Delta$

But increasingly important for weaker couplings or lower densities

Not able to get results for $1/(pFa_0) \lesssim -2$
- effective potential nonanalytic in $\phi$ for small gaps
→ expansion of effective action breaks down

For parameters corresponding to neutron matter
- gap comparable to $\epsilon_F$ ($\sim 30$ MeV)
- more realistic treatments give $\Delta \sim 5$ MeV
→ need to keep higher-order terms in effective-range expansion
Future work

Long list of “things to do” including:

Renormalisation of boson kinetic mass, $Z_m$
→ scaling analysis of boson loops for small gaps

Complete analysis of current ansatz for $\Gamma$
– running of fermion renormalisation factors, $Z_{\psi,M}$
  and “Yukawa” coupling, $Z_g$

Adding momentum-dependent interactions (effective range)
→ more realistic interaction strength at Fermi surface

Treating explicitly particle-hole channels (RPA phonons)
– important physics [Schwenk]
– remove Fierz ambiguity in bosonisation
  [Jaeckel and Wetterich, hep-ph/0207094]

Adding three-body forces