Questions refer to material from lectures 7 to 12. They cover quantum scattering with applications to NN systems, general features of nuclear reactions, resonances, and direct reactions.

1. Write down the differential cross section for a system where only the S-wave and P-wave phase shifts are nonzero. For nucleon-nucleon scattering at a centre-of-mass energy of 25 MeV, take the S-wave phase shift to be $\delta_0 = 40°$ and the P-wave shift to be $\delta_1 = -10°$. Assume that all other phase shifts can be neglected. Plot the resulting differential cross section as a function of angle.

2. Consider a system of two spin-up protons.

   (a) Explain why their scattering vanishes at low energies.

   (b) Show that their scattering cross section is symmetric between forward and backward angles (i.e. under the replacement $\theta \to \pi - \theta$). Give a physical explanation for this result.

3. A particle moves in the potential $V(\mathbf{r})$. Its wave function $\Psi(\mathbf{r}, t)$ satisfies the time-dependent Schrödinger equation,

$$\mathrm{i}\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi + V(\mathbf{r})\Psi.$$

   (a) Show that the usual probability density $\rho = \Psi^*\Psi$ and the current density

$$\mathbf{j} = \frac{\hbar}{2\mathrm{i}M}\left[\Psi^*\nabla\Psi - (\nabla\Psi^*)\Psi\right]$$

   satisfy the continuity equation

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t},$$

   provided $V(\mathbf{r})$ is real. Give a physical interpretation of this result.

   [*Hint: You will also need the complex conjugate of the Schrödinger equation.*]

   (b) Evaluate the current density for the plane wave

$$\psi(\mathbf{r}) = A\,\mathrm{e}^{\mathrm{i}kz},$$

   where A is a complex constant.

   (c) Evaluate the current density at large r for the outgoing wave

$$\psi(\mathbf{r}) = f(\theta, \phi)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r}.$$

   Hence show that the corresponding total probability current through a large sphere is independent of the radius r.

4. Consider scattering by a (real) central potential. Written in terms of an incoming plane wave and an outgoing scattered wave, the asymptotic form of the resulting wave function has the partial-wave expansion

$$\mathrm{e}^{\mathrm{i}kz} + f(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r} = \frac{1}{2\mathrm{i}kr}\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}\Big[(1+f_l)\mathrm{e}^{\mathrm{i}kr} - (-1)^l\mathrm{e}^{-\mathrm{i}kr}\Big]Y_{l0}(\theta),$$

where

$$f(\theta) = \frac{1}{2\mathrm{i}k}\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}\,f_l\,Y_{l0}(\theta).$$

The same wave function expressed in terms of the asymptotic forms of the solutions of the radial Schrödinger equation ($\sin(kr+\delta_l-l\pi/2)$) is

$$\mathrm{e}^{\mathrm{i}kz} + f(\theta)\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r} = \frac{1}{2\mathrm{i}kr}\sum_{l=0}^{\infty} b_l\Big[(-\mathrm{i})^l\mathrm{e}^{\mathrm{i}\delta_l}\mathrm{e}^{\mathrm{i}kr} - \mathrm{i}^l\mathrm{e}^{-\mathrm{i}\delta_l}\mathrm{e}^{-\mathrm{i}kr}\Big]Y_{l0}(\theta).$$

By equating the coefficients of $\mathrm{e}^{-\mathrm{i}kr}$ and $\mathrm{e}^{\mathrm{i}kr}$ in each term of these expressions, show that

$$\begin{aligned} b_l &= \sqrt{4\pi(2l+1)}\,\mathrm{i}^l\mathrm{e}^{\mathrm{i}\delta_l}, \\ f_l &= \mathrm{e}^{2\mathrm{i}\delta_l} - 1 = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\delta_l}\sin\delta_l. \end{aligned}$$

Give a physical explanation for the fact that $|1+f_l| = 1$ here.

5. (a) In lectures we considered a model of two nucleons interacting through a square well potential of radius R and depth V_0. The resulting reduced wave function for a positive-energy S-wave solution has the form

$$u(r) = \begin{cases} A\sin(Kr) & \text{for } r < R \\ B\sin(kr+\delta) & \text{for } r > R \end{cases},$$

where $k = \sqrt{ME}/\hbar$ and $K = \sqrt{M(V_0+E)}/\hbar$. By matching u and u' at $r = R$, show that

$$\frac{1}{K}\,\tan(KR) = \frac{1}{k}\,\tan(kR+\delta).$$

Hence show that the phase shift δ is given by

$$\tan\delta = \frac{k\tan(KR) - K\tan(kR)}{K + k\tan(KR)\tan(kR)}.$$

[*Hint: You may find the trig identity*

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

helpful.]

(b) We also found that a square-well potential with a radius $R = 1.4$ fm and depth $V_0 = 69$ MeV could bind two nucleons to form a deuteron with the observed binding energy. Now consider S-wave scattering by the same potential and find the phase shifts at the lab-frame energies: 100 keV, 1 MeV, 10 MeV, and 100 MeV. [Think carefully about which quadrant δ should be in.]

(c) Now consider a shallower potential with the same radius and $V_0 = 51$ MeV. Find the phase shifts it produces at the same three energies.

(d) Compare your results with the actual ${}^3S_1$ and ${}^1S_0$ phase shifts deduced from experiment.

[The results of partial-wave analyses by the Nijmegen group can be found on their website, NN-OnLine: `http://nn-online.org/`. Click on the link "NN interaction" and you will get various options for calculating or plotting phase shifts and observables. Their standard fit to the data is the one labelled "PWA93".]

(e) By considering the limit of $\tan\delta/k$ as $k \to 0$, determine the S-wave scattering lengths for the potentials in (b) and (c). Compare your results with the experimental values and with $1/\gamma$ for the deuteron.

6. Find plots of the phase shifts for ${}^1D_2$ np and pp scattering and np ${}^3D_2$ scattering. [For example, download these from NN-Online.] By comparing their forms, deduce what you can from them about the interactions involved. Which of these phases would you expect D-wave nn scattering to resemble most?

7. For a system with only elastic S-wave scattering, the scattering amplitude has the form

$$f(\theta) = \frac{1}{k}\, \mathrm{e}^{\mathrm{i}\delta} \sin\delta.$$

Write down the corresponding total cross section σ and verify that it satisfies the optical theorem,

$$\mathrm{Im}[f(0)] = \frac{k}{4\pi}\, \sigma.$$

8. In lectures we studied resonances by assuming that the logarithmic derivative of the wave function was a smoothly varying function of energy. Near an energy E_0 where the real part of this derivative vanishes, this allows us to write

$$\frac{u'(0)}{u(0)} = \mathrm{i}k\,\frac{\eta_0+1}{\eta_0-1} \simeq -\,\frac{E-E_0+\mathrm{i}\,\dfrac{\Gamma_{\rm re}}{2}}{\beta},$$

where η_0 is the amplitude of the outgoing wave and β is a real constant.

(a) By solving this equation for η_0, show that

$$\eta_0 \simeq \frac{E-E_0-\mathrm{i}\,\dfrac{\Gamma_{\rm el}}{2}+\mathrm{i}\,\dfrac{\Gamma_{\rm re}}{2}}{E-E_0+\mathrm{i}\,\dfrac{\Gamma_{\rm el}}{2}+\mathrm{i}\,\dfrac{\Gamma_{\rm re}}{2}},$$

where $\Gamma_{\rm el} = 2\beta k$.

(b) Hence show that, near the resonance, the elastic and reaction cross sections have the forms

$$\sigma_{\rm el} \simeq \frac{\pi}{k^2}\,\frac{\Gamma_{\rm el}^2}{(E-E_0)^2+\dfrac{\Gamma^2}{4}},$$

$$\sigma_{\rm re} \simeq \frac{\pi}{k^2}\,\frac{\Gamma_{\rm el}\Gamma_{\rm re}}{(E-E_0)^2+\dfrac{\Gamma^2}{4}},$$

where $\Gamma = \Gamma_{\rm el}+\Gamma_{\rm re}$.

(c) Write down the total cross section $\sigma_{\rm tot}$ for this resonance and show that at $E = E_0$, the resonance peak,

$$\sigma_{\rm tot}^2 = \frac{4\pi}{k^2}\,\sigma_{\rm el}.$$

9. The Breit-Wigner formula for a single-resonance cross section is

$$\sigma_{\alpha\beta} = \frac{\pi}{k^2}\, g \, \frac{\Gamma_\alpha \Gamma_\beta}{(E - E_0)^2 + \dfrac{\Gamma^2}{4}},$$

where, for incoming particles with spins J_A and J_B producing a resonance with spin J, the angular-momentum coupling factor is

$$g = \frac{2J+1}{(2J_A+1)(2J_B+1)}.$$

Scattering of low-energy neutrons from ^{235}U (which has $J_B = 7/2$) shows a strong $J = 3$ resonance at $E_0 = 0.29$ eV, with a width of $\Gamma = 0.135$ eV. This resonance can decay by neutron emission, photon emission or fission. The dominant process is neutron-induced fission, and this has a cross section of $\sigma_f = 200$ b at the peak of the resonance. Radiative capture has a peak cross section of $\sigma_\gamma = 70$ b, while the cross section for elastic neutron scattering is too small to be measured, $\sigma_n \ll 1$ b.

(a) Use this information to determine the partial widths for each of these channels.

(b) Hence calculate the elastic cross section at the peak.

10. The differential cross section for the stripping reaction $^{40}\mathrm{Ca}(d,p)\,^{41}\mathrm{Ca}$ has been measured at a deuteron energy $E_{\mathrm{lab}} = 12.0$ MeV [K. K. Seth *et al.*, Nucl. Phys. A **140** (1970) 577]. The cross section for production of the excited state of ^{41}Ca at $E_x = 2.471$ MeV shows peaks at $\theta \simeq 13°$, $45°$ and $77°$.

(a) Find the initial and final energies of this system in the centre-of-mass frame. Hence calculate the momentum transfers (in fm^{-1}) at the three peaks.

(b) Use the PWBA to verify that the assignment of $l = 1$ for the neutron in this state of ^{41}Ca is consistent. Estimate the radius at which the reaction occurs and compare your answer with $1.2\,A^{1/3}$ fm for the ^{40}Ca core. Check also that $l = 3$, the other possible value for a nucleon in the 5th shell, is implausible.

[From Krane, Appendix C, the masses of the isotopes involved are:

	M (u)
^{1}H	1.007825
^{2}H	2.014102
^{40}Ca	39.962591
^{41}Ca	40.962278

The spherical Bessel function $j_1(x)$ has maxima at $x = 2.08$, 5.94 and 9.21, while $j_3(x)$ has these at $x = 4.51$, 8.58 and 11.97.]