Questions refer to material from lectures 1 to 6. They cover symmetries and other general features of nuclear forces, and properties of the deuteron.

1. (a) What are the allowed values of the orbital angular momentum L for two neutrons with spin $S = 0$?

   (b) For two nucleons with orbital angular momentum L, spin S and isospin T, show that the sum $L + S + T$ must be odd.

   (c) A proton and a neutron are in a ${}^3G_3$ state. Are there other states with same total J and parity that could mix with this one? Are there pp and nn states with the same quantum numbers?

2. Find the possible values of the total isospin for a system consisting of a pion and a Δ baryon. What are the possible total charges for the states corresponding to the largest value of T? Comment on how you might expect $\pi\Delta$ scattering in a state with the largest value of T to be different from that in ones with smaller values of T.

3. For a system of two spin-$\frac{1}{2}$ particles, find the eigenvalues of $\widehat{\mathbf{s}}^{(1)} \cdot \widehat{\mathbf{s}}^{(2)}$. where $\widehat{\mathbf{s}}^{(i)}$ is the vector spin operator for particle i.

   [*Hint: start by squaring* $\widehat{\mathbf{S}} = \widehat{\mathbf{s}}^{(1)} + \widehat{\mathbf{s}}^{(2)}$.]

4. For a system of two nucleons, find the eigenvalues of $\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}$.

   [*Hint: use* $\widehat{\mathbf{t}}^{(i)} = \frac{1}{2}\boldsymbol{\tau}^{(i)}$ *and your answer to the previous question.*]

5. The total magnetic moment of a proton and a neutron is $\mu_S = 0.8798\,\mu_N$ if they are in a ${}^3S_1$ state, and $\mu_D = 0.3101\,\mu_N$ if they are in a ${}^3D_1$ state. Use these results and the observed deuteron magnetic moment, $\mu_d = 0.8574\,\mu_N$, to estimate the D-state probability of the deuteron. (You may assume that the magnetic moment operator does not couple S and D states.) Comment on whether you expect this to be an accurate estimate.

6. In lectures, we studied a simple model for the deuteron as an S-wave bound state in a square well potential of radius R. The resulting reduced wave function had the form

$$u(r) = \begin{cases} A\sin(Kr) & \text{for } r < R \\ B\mathrm{e}^{-\gamma r} & \text{for } r > R \end{cases},$$

   where the deuteron binding energy gives $\gamma = 0.23\ \mathrm{fm}^{-1}$. By matching the pieces smoothly at $r = R$ we obtained $K = 1.25\ \mathrm{fm}^{-1}$ for a well of radius $R = 1.4$ fm.

   (a) First use the matching condition to determine B/A, then use the normalisation condition to determine A.

(b) What is the probability for finding the proton and neutron beyond the radius R of the interaction?

(c) Calculate the charge radius of the deuteron in this model and compare your result with the experimental value.

7. Show that (at least for $r \neq 0$)

$$(\boldsymbol{\sigma}^{(1)} \cdot \nabla)(\boldsymbol{\sigma}^{(2)} \cdot \nabla)\, \frac{\mathrm{e}^{-r/R}}{r} = \left[(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r}) \left(\frac{1}{R^2\, r^3} + \frac{3}{R\, r^4} + \frac{3}{r^5}\right) - \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \left(\frac{1}{R\, r^2} + \frac{1}{r^3}\right)\right] \mathrm{e}^{-r/R}.$$

You may use the results

$$\nabla f(r) = \mathbf{r}\, \frac{1}{r}\, \frac{\mathrm{d}f}{\mathrm{d}r}, \quad \text{and} \quad \nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}.$$

Show also that this can be rewritten in the form

$$(\boldsymbol{\sigma}^{(1)} \cdot \nabla)(\boldsymbol{\sigma}^{(2)} \cdot \nabla)\, \frac{\mathrm{e}^{-r/R}}{r} = \frac{1}{3R^2} \left[\widehat{S}_{12} \left(1 + \frac{3R}{r} + \frac{3R^2}{r^2}\right) + \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}\right] \frac{\mathrm{e}^{-r/R}}{r},$$

where the tensor operator is

$$\widehat{S}_{12} = \frac{3}{r^2}\, (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r}) - \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}.$$

8. Evaluate the expectation value of $\widehat{S}_{12}$ in the state ψ_{11} where the two nucleons have total spin $S = 1$ and $M_S = +1$. Show that your answer is proportional to $Y_{20}(\theta, \phi)$.