1. The scattering amplitude for this system is

$$f(\theta) = \frac{1}{k}\left[\sqrt{4\pi}\,\mathrm{e}^{\mathrm{i}\delta_0}\sin\delta_0\,Y_{00}(\theta) + \sqrt{12\pi}\,\mathrm{e}^{\mathrm{i}\delta_1}\sin\delta_1\,Y_{10}(\theta)\right].$$

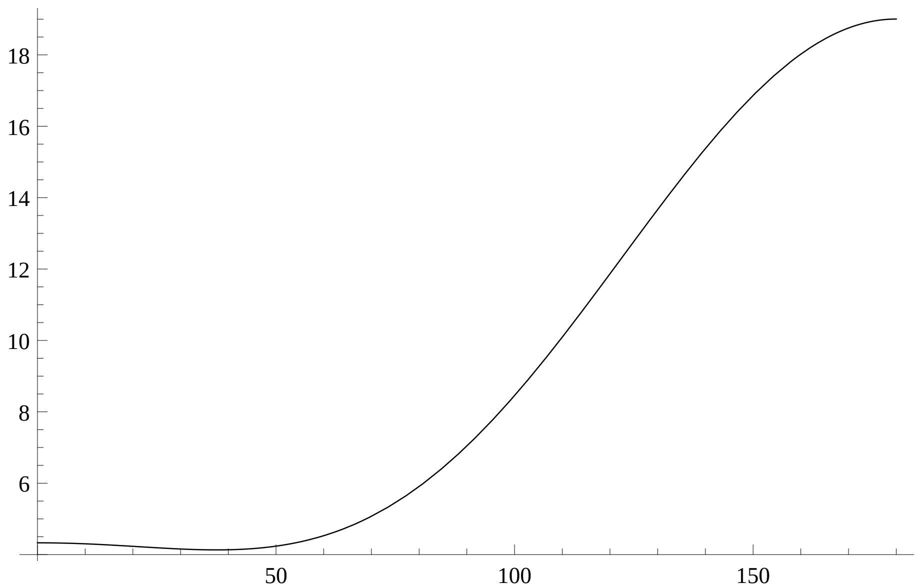
The relevant spherical harmonics are

$$Y_{00}(\theta) = \frac{1}{\sqrt{4\pi}}, \qquad Y_{10}(\theta) = \sqrt{\frac{3}{4\pi}}\cos\theta.$$

Hence the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2 = \frac{1}{k^2}\Big[\sin^2\delta_0 + 9\sin^2\delta_1\cos^2\theta + 6\cos(\delta_0-\delta_1)\sin\delta_0\sin\delta_1\cos\theta\Big].$$

For NN scattering at a centre-of-mass energy $E = 25$ MeV, the wavenumber is $k = \sqrt{ME}/\hbar = 0.77$ fm and so $1/k^2 = 17$ mb. For the phases in the question, the dependence of the differential cross section (in mb/sr) on θ (in degreees) has the form



2. (a) Antisymmetry requires the two protons to have odd orbital angular momentum and hence $L \geq 1$. Since phase shifts vanish at low energies as $\delta_L \propto k^{2L+1}$, the total cross section will have the form

$$\sigma \simeq \frac{4\pi}{k^2}\sin^2\delta_1 \propto k^4,$$

which vanishes as $k \to 0$. (Waves with $L > 1$ will give contributions that vanish even faster.)

(b) More generally, the scattering amplitude for the protons is

$$f(\theta) = \frac{1}{k} \sum_{\text{odd } L} \sqrt{4\pi(2L+1)}\, \mathrm{e}^{\mathrm{i}\delta_L} \sin\delta_L\, Y_{L0}(\theta).$$

Since $\theta \to \pi - \theta$ is just parity reversal for functions that are independent of ϕ, all the $Y_{L0}(\theta)$ for odd L are odd under this reflection. (Alternatively, you could use the fact that they are all odd polynomials in $\cos\theta$.) The differential cross section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2,$$

is therefore even under $\theta \to \pi - \theta$. This symmetry is to be expected: the two protons are indistinguishable.

3. (a) The divergence of $\mathbf{j}$ is

$$\nabla \cdot \mathbf{j} = \frac{\hbar}{2\mathrm{i}M}\big[\Psi^* \nabla^2 \Psi - (\nabla^2 \Psi^*)\Psi\big].$$

We can substitute for $\nabla^2\Psi$ and $\nabla^2\Psi^*$ using the Schrödinger equation and its conjugate,

$$-\mathrm{i}\hbar\, \frac{\partial \Psi^*}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi^* + \Psi^* V^*.$$

This gives

$$\nabla \cdot \mathbf{j} = -\Psi^* \left(\frac{\partial \Psi}{\partial t} + \frac{\mathrm{i}}{\hbar}\, V\Psi\right) - \left(\frac{\partial \Psi^*}{\partial t} - \frac{\mathrm{i}}{\hbar}\, \Psi^* V^*\right)\Psi,$$

which reduces to the continuity equation

$$\nabla \cdot \mathbf{j} = -\frac{\partial}{\partial t}\, \Psi^*\Psi = -\frac{\partial \rho}{\partial t},$$

provided V is real.

This shows that probability is conserved: the only way that the probability of finding a particle inside some region can change is if there is a flux through the surface surrounding the region.

(b) The current density for this wave is

$$j_z = |A|^2 \frac{\hbar k}{M},$$

which is just the product of the probability density $|A|^2$ and the clasical speed of the particle (p/M).

(c) The gradient operator in spherical polar coordinates is

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

Only the term in $\nabla\psi$ where $\partial/\partial r$ acts on the exponential gives a result that falls off like $1/r$; everything else goes like $1/r^2$ and so can be neglected for large enough r. This leaves

$$\nabla\psi \to \frac{\mathrm{i}\, k\, \mathrm{e}^{\mathrm{i}kr}}{r}\, f(\theta, \phi)\, \mathbf{e}_r.$$

With this, we get the current density

$$\mathbf{j} \to \frac{\hbar k}{M\,r^2}\,|f(\theta,\phi)|^2 \mathbf{e}_r.$$

This is purely radial and so the total probability flowing through a sphere of radius r is

$$I = r^2 \int |\mathbf{j}|\,\mathrm{d}\Omega = \frac{\hbar k}{M} \int |f(\theta,\phi)|^2\,\mathrm{d}\Omega,$$

which is independent of r.

4. From the coefficients of the incoming wave $\mathrm{e}^{-\mathrm{i}kr}$ in the two expressions, we get

$$(-1)^l \sqrt{4\pi(2l+1)} = \mathrm{i}^l\, b_l\, \mathrm{e}^{-\mathrm{i}\delta_l},$$

and so

$$b_l = \mathrm{i}^l \sqrt{4\pi(2l+1)}\,\mathrm{e}^{\mathrm{i}\delta_l}.$$

Similarly, from the outgoing waves, we get

$$\sqrt{4\pi(2l+1)}(1+f_l) = (-\mathrm{i})^l\, b_l\, \mathrm{e}^{\mathrm{i}\delta_l},$$

and so

$$1 + f_l = \frac{(-\mathrm{i})^l\, b_l\, \mathrm{e}^{\mathrm{i}\delta_l}}{\sqrt{4\pi(2l+1)}} = \mathrm{e}^{2\mathrm{i}\delta_l}.$$

Hence the amplitudes of the scattered waves are

$$f_l = \mathrm{e}^{2\mathrm{i}\delta_l} - 1 = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\delta_l}\sin\delta_l.$$

This shows that probability is conserved in each partial wave.

5. (a) The matching conditions are

$$\begin{aligned} u(R) = A\sin KR &= B\sin(kR+\delta), \\ u'(R) = KA\cos KR &= kB\cos(kR+\delta). \end{aligned}$$

Taking the ratio of these gives

$$\frac{1}{K}\tan(KR) = \frac{1}{k}\tan(kR+\delta).$$

With the help of the given identity, this becomes

$$\frac{1}{K}\tan(KR) = \frac{1}{k}\,\frac{\tan(kR)+\tan\delta}{1-\tan(kR)\tan\delta},$$

which can be rewritten as a linear equation $\tan\delta$. Solving this gives

$$\tan\delta = \frac{k\tan(KR) - K\tan(kR)}{K + k\tan(KR)\tan(kR)}.$$

(b) For two particles of equal mass, the centre-of-mass energy is $E = E_{\rm lab}/2$. From this we can determine k and K for each energy and hence obtain δ.

$E_{\rm lab}$ (MeV)	$\delta_{\rm model}$	$\delta_{\rm expt}$
0.10	170°	169°
1.0	150°	148°
10	110°	103°
100	59°	43°

Here the "experimental" values are from the Nijmegen PWA93 $^3S_1$ wave. (This is the channel which has a bound state, the deuteron.)

(c) In the same way, we get the phases for the weaker potential.

$E_{\rm lab}$ (MeV)	$\delta_{\rm model}$	$\delta_{\rm expt}$
0.10	37°	38°
1.0	64°	62°
10	69°	60°
100	46°	27°

This time the "experimental" values are from the Nijmegen PWA93 $^1S_0$ wave.

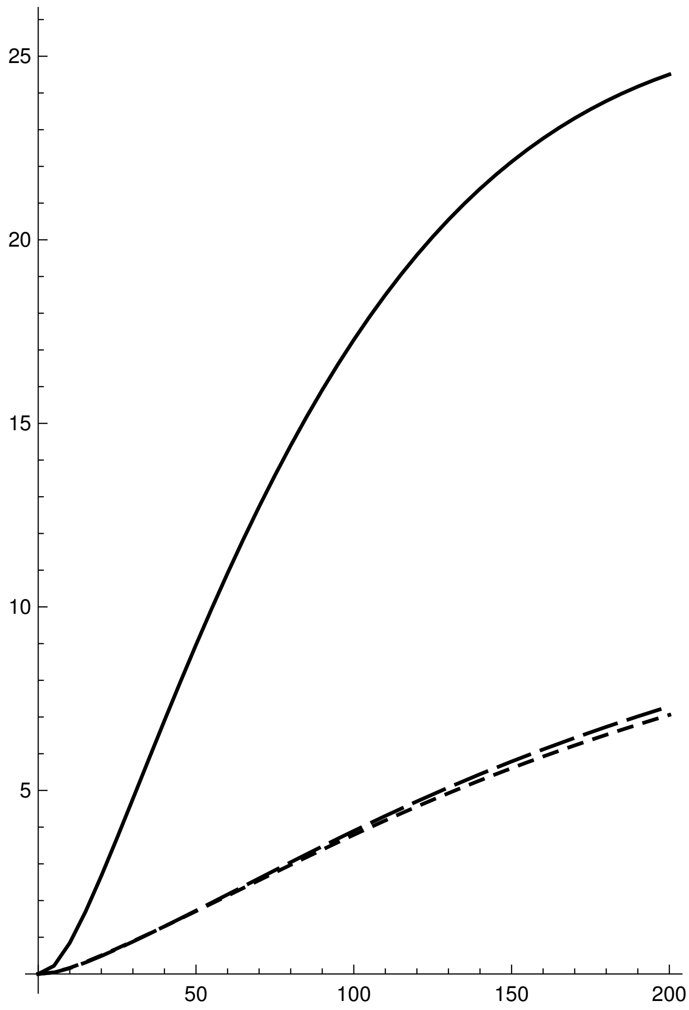
(d) The agreement is remarkably good, given the simple model we have used. In each case, I have adjusted only one parameter, V_0, to fit one observable, either the deuteron binding energy or the singlet scattering length. In these channels, NN scattering at low energies is dominated by the existence (or near existence) of a bound state, and is relatively insensitive to the detailed shape of the potential, especially at energies below about 10 MeV).

(e) As $k \to 0$, we can use $\tan x \simeq x$ to get

$$a = -\lim_{k\to 0} \frac{\tan\delta}{k} = -\frac{1}{K_0}\big[\tan(K_0 R) - K_0 R\big],$$

where $K_0 = \sqrt{MV_0}/\hbar$. For the stronger potential of part (b), this gives $a = 5.1$ fm. For the weaker one, it gives $a = -21.1$ fm. For comparison, the observed triplet and singlet scattering lengths are $a_t = 5.4$ fm and $a_s = -23.7$ fm. In the zero-range limit, the triplet length would be exactly equal to the deuteron decay length, $1/\gamma = 4.4$ fm. The difference is a result of the finite effective range. Our stronger potential has too short a range, and so its scattering length lies between these values. The weaker potential gives a scattering length that agrees with singlet length (within the accuracy of the value for V_0 given in the question).

6. The phase shifts from the Nijmegen PWA93 are shown in the figure (plotted in degrees against lab-frame energy in MeV).



Here the phase for $^3D_2$ np scattering is given by the solid line, $^1D_2$ np by the long-dashed line, and $^1D_2$ pp by the short-dashed line. One immediate observation is that the scattering is attractive in all these channels (the phase shifts are positive). We can also see that the forces depend on spin and/or isospin (the phase for the channel with $S = 1$ and $T = 0$ is very different from those for the channels with $S = 0$ and $T = 1$). Isospin is clearly a good but not perfect symmetry (the phases for the two $^1D_2$ channels are very similar but not identical).

In waves like these with nonzero L, we would expect the scattering to be more sensitive to long-ranged pion-exchange forces. The largest isospin-breaking effect in these is the difference between the masses of the neutral and charged pions. The leads to interactions that break isospin symmetry but still respect charge symmetry, and so we would expect the $^1D_2$ nn phase shift (could it ever be measured) to be most similar to the pp one.

7. The elastic cross section is for S-wave scattering is

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta.$$

The imaginary part of the forward scattering amplitude is thus

$$\mathrm{Im}[f(0)] = \frac{1}{k}\sin^2\delta = \frac{k}{4\pi}\,\sigma.$$

8. (a) The equation in the question can be rewritten as

$$\frac{\mathrm{i}\Gamma_{\rm el}}{2}\,\frac{\eta_0+1}{\eta_0-1} \simeq -\left[E - E_0 + \mathrm{i}\,\frac{\Gamma_{\rm re}}{2}\right].$$

Multiply this by $\eta_0 - 1$ and solve the resulting linear equation for η_0.

(b) The elastic cross section is

$$\sigma_{\rm el} = \frac{\pi}{k^2}\,|1-\eta_0|^2,$$

and the reaction cross section is

$$\sigma_{\rm re} = \frac{\pi}{k^2}\,\left(1 - |\eta_0|^2\right).$$

Substitute the answer from part (a) into these expressions.

(c) At the resonance peak the terms involving $E - E_0$ vanish, leaving

$$\sigma_{\rm el} \simeq \frac{4\pi}{k^2}\,\frac{\Gamma_{\rm el}^2}{\Gamma^2},$$

$$\sigma_{\rm re} \simeq \frac{4\pi}{k^2}\,\frac{\Gamma_{\rm el}\Gamma_{\rm re}}{\Gamma^2}.$$

The total cross section is

$$\sigma_{\rm tot} = \sigma_{\rm el} + \sigma_{\rm re} = \frac{4\pi}{k^2}\,\frac{\Gamma_{\rm el}}{\Gamma},$$

and hence its square is

$$\sigma_{\rm tot}^2 = \frac{4\pi}{k^2}\,\sigma_{\rm el}.$$

9. (a) The peak cross sections for neutron-induced fission and radiative capture are

$$\sigma_f = \frac{4\pi}{k^2}\,g\,\frac{\Gamma_n\Gamma_f}{\Gamma^2}, \qquad \sigma_\gamma = \frac{4\pi}{k^2}\,g\,\frac{\Gamma_n\Gamma_\gamma}{\Gamma^2},$$

where Γ_n, Γ_f and Γ_γ are the partial widths for decay of the resonance by neutron emission, fission and photon emission. The total width of the resonance is, neglecting the very small contribution from neutron emission, $\Gamma \simeq \Gamma_f + \Gamma_\gamma$. For this resonance at $E = 0.29 \times 10^{-6}$ MeV, we have $k = \sqrt{ME}/\hbar c$ $= 1.2 \times 10^{-4}$ fm^{-1} and $g = 7/16$. From the reaction cross sections we get

$$\Gamma_n\Gamma_f = \frac{k^2\sigma_f}{4\pi g}\,\Gamma^2 = 9.5 \times 10^{-7}\ \mathrm{eV}^2,$$

$$\Gamma_n\Gamma_\gamma = \frac{k^2\sigma_\gamma}{4\pi g}\,\Gamma^2 = 3.3 \times 10^{-7}\ \mathrm{eV}^2.$$

Adding these gives

$$\Gamma_n = 9.5 \times 10^{-6} \text{ eV},$$

and hence

$$\Gamma_f = 0.100 \text{ eV}, \qquad \Gamma_\gamma = 0.035 \text{ eV}.$$

(b) The elastic cross section is

$$\sigma_n = \frac{4\pi}{k^2}\, g\, \frac{\Gamma_n^2}{\Gamma^2} = 1.9 \text{ fm}^2 = 19 \text{ mb}.$$

10. (a) The initial energy in the centre-of-mass frame of this system is

$$E = \frac{40}{42} E_{\text{lab}} = 11.4 \text{ MeV}.$$

From the given masses and excitation energy, the energy released is

$$\begin{aligned} Q &= \left[M(^2\text{H}) + M(^{40}\text{Ca}) - M(^1\text{H}) - M(^{41}\text{Ca})\right] c^2 - E_x \\ &= 6.59 \times 10^{-3} \text{ u}\, c^2 - 2.47 \text{ MeV} = 3.67 \text{ MeV} \end{aligned}$$

(using 1 u $= 931.5$ MeV$/c^2$), and hence the final energy is

$$E' = E + Q = 15.1 \text{ MeV}.$$

Combining these energies with the reduced masses, $\mu = 1770$ MeV$/c^2$ and $\mu' = 910$ MeV$/c^2$, we get the initial and final wave numbers, $k = 1.00$ fm^{-1} and $k' = 0.83$ fm^{-1}. The momentum transfer for a reaction where the outgoing particles are at an angle θ to the beam is

$$q = \sqrt{k^2 + k'^2 - 2kk' \cos\theta}.$$

At the angles of the three peaks, this gives $q = 0.27$, 0.72 and 1.15 fm^{-1}.

(b) In the PWBA, we expect the differential cross section to have the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |j_l(qr)|^2.$$

If we assume that $l = 1$ then the ratios of the values of q from part (a) are consistent with the ratios of the values of $x = qR$ at which $j_1(x)$ has peaks. Alternatively if we use these to deduce a radius at which the reaction takes place, we get $R = 7.8$, 8.2 and 8.0 fm, which are compatible with a single radius of about 8 fm. For comparison, the ^{40}Ca core has a radius of about 4.1 fm. Our deduced radius is reasonable for a reaction that involves the tails of nuclear wave functions.

In contrast, if we assume $l = 3$, then the peaks do not follow the pattern of $j_3(qR)$. For example, if we try to deduce a reaction radius we get different values from each: $R = 16.9$, 11.9 and 10.4 fm. Moreover the first of these is implausibly large, lying more than 10 fm outside the ^{40}Ca core.