1. (a) The overall state of two identical fermions must be antisymmetric under exchange. Since the $S = 0$ spin state is antisymmetric, the orbital wave function must be symmetric and hence L must be even.

   (b) Perhaps the easiest way to do this is to collect the phase factors that are generated by exchanging the two particles. For the orbital wave function, exchange is like parity reversal and so it gives a factor $(-1)^L$. The antisymmetry of the $S = 0$ state (and symmetry of $S = 1$) can be represented by a factor $(-1)^{S+1}$. The isospin states are similar, giving a factor $(-1)^{T+1}$. Putting these together, antisymmetry requires $(-1)^{L+S+T+2} = -1$ and hence $L + S + T$ must be odd.

   (c) The spectroscopic notation ${}^{2S+1}L_J = {}^3G_3$ corresponds to the quantum numbers $L = 4$, $S = 1$ and $J = 3$. Since L is even, the parity is too. The neighbouring waves with the same parity are $L = 2$ and $L = 6$. However only $L = 2$ can combine with $S = 1$ to give $J = 3$. The ${}^3G_3$ wave can therefore mix with ${}^3D_3$. Since the orbital and spin states are both symmetric, the isospin state must be antisymmetric: $T = 0$. There are thus only pn states with these quantum numbers.

2. The pion has isospin $T_\pi = 1$ and the Δ has $T_\Delta = 3/2$. Adding these is just like adding angular momenta and so the possible values for the total are

$$T = |T_\pi - T_\Delta|,\ \ldots,\ T_\pi + T_\Delta = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}.$$

For $T = 5/2$, the possible eigenvalues of T_3 are

$$T_3 = +T,\ T-1,\ \ldots,\ -T = +\frac{5}{2},\ +\frac{3}{2},\ \ldots,\ -\frac{5}{2}.$$

Since the baryon number of the system is $B = 1$, the charges of these states are

$$Q = \frac{1}{2}B + T_3 = +3, +2,\ \ldots,\ -2.$$

The states with $T = 1/2$ and $3/2$ can couple strongly to the nucleon, Δ, and their excited states. In contrast, baryons with $T = 5/2$ do not appear in the quark model and they have never been conclusively observed. We do not expect to see any "resonances" in this channel, unlike the other two.

3. Squaring $\widehat{\mathbf{S}} = \widehat{\mathbf{s}}^{(1)} + \widehat{\mathbf{s}}^{(2)}$ and rearranging (as we have done before in the context of spin-orbit interactions), we can express the product in terms of squares of angular momenta,
$$\widehat{\mathbf{s}}^{(1)} \cdot \widehat{\mathbf{s}}^{(2)} = \frac{1}{2}\left[\widehat{\mathbf{S}}^2 - \left(\widehat{\mathbf{s}}^{(1)}\right)^2 - \left(\widehat{\mathbf{s}}^{(2)}\right)^2\right].$$
We can now write down the eigenvalues of the product in terms of the usual eigenvalues of the squares,
$$\begin{aligned}\widehat{\mathbf{s}}^{(1)} \cdot \widehat{\mathbf{s}}^{(2)} &= \frac{\hbar^2}{2}\left[S(S+1) - \frac{3}{4} - \frac{3}{4}\right] \\ &= \begin{cases} -\frac{3}{4}\hbar^2 & \text{for } S = 0 \\ +\frac{1}{4}\hbar^2 & \text{for } S = 1 \end{cases}.\end{aligned}$$

4. The isospin operators work just like spins and so we can use the results of the previous question (without the factors of $\hbar$) to get
$$\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)} = 4\,\widehat{\mathbf{t}}^{(1)} \cdot \widehat{\mathbf{t}}^{(2)} = \begin{cases} -3 & \text{for } T = 0 \\ +1 & \text{for } T = 1 \end{cases}.$$

5. The expectation value of magnetic moment operator has the form
$$\langle \widehat{\mu} \rangle = a_S^2\, \mu_S + a_D^2\, \mu_D,$$
where $a_{S,D}$ are the amplitudes of the (normalised) S- and D-wave components of the deuteron. There is no term containing $a_S\, a_D$ since the operator is built out of $\widehat{\mathbf{L}}$, $\widehat{\mathbf{S}}_p$ and $\widehat{\mathbf{S}}_n$, all of which commute with $|\widehat{\mathbf{L}}|^2$, and so it does not connect waves with different values of L. Normalisation of the deuteron wave function requires
$$a_S^2 + a_D^2 = 1,$$
and so the D-state probability is
$$P_D = a_D^2 = 1 - a_S^2.$$
Hence we can write
$$\mu_d = (1 - P_D)\mu_S + P_D\, \mu_D.$$
Solving for P_D we get
$$P_D = 0.039.$$
This is the right order of magnitude but it is not expected to be accurate as there are other contributions to the deuteron magnetic moment. These come from coupling of the magnetic field to the two nucleons while they are interacting, for example by exchanging a charged pion.

6. (a) Matching the wave function at $r = R$ gives

$$A\sin(KR) = B\mathrm{e}^{-\gamma R},$$

and hence

$$B/A = \mathrm{e}^{\gamma R}\sin(KR) = 1.36$$

Normalisation requires

$$\begin{aligned} 1 = \int_0^\infty u(r)^2\,\mathrm{d}r &= A^2\int_0^R \sin^2(Kr)\,\mathrm{d}r + B^2\int_R^\infty \mathrm{e}^{-2\gamma r}\mathrm{d}r \\ &= 2.88\,A^2, \end{aligned}$$

and hence $A = 0.59$ (and $B = 0.80$).

(b) The probability of finding the nucleons outside the range of the force is just the second term in the integral above,

$$P(r > R) = B^2\int_R^\infty \mathrm{e}^{-2\gamma r}\mathrm{d}r = 0.73.$$

(c) The mean-square separation of the proton and neutron is

$$\langle r^2\rangle = \int_0^\infty r^2 u(r)^2\,\mathrm{d}r = A^2\int_0^R r^2\sin^2(Kr)\,\mathrm{d}r + B^2\int_R^\infty r^2\mathrm{e}^{-2\gamma r}\mathrm{d}r.$$

After integrating by parts, this gives

$$\langle r^2\rangle = 13.1\ \mathrm{fm}^2,$$

and hence an rms separation of $\sqrt{\langle r^2\rangle} = 3.6$ fm. For the charge radius we need the rms distance of the proton from the centre-of-mass, which is half the pn separation:

$$\sqrt{\langle r_p^2\rangle} = 1.8\ \mathrm{fm}.$$

This is a bit smaller than the radius of 1.97 fm deduced from experiment, but the square well is a highly simplified model for the potential.

7. The easiest way to get from the first expression to the second is to add and subtract the term $\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}/3R^2r$ inside the square bracket.

8. The $M_S = +1$ state is just the product of two spin-up spinors:

$$\psi_{11} = \alpha(1)\alpha(2).$$

The expectation value of $\widehat{S}_{12}$ in this state is

$$\begin{aligned}\psi_{11}^\dagger \widehat{S}_{12} \psi_{11} = \frac{3}{r^2} &\left(\alpha(1)^\dagger \boldsymbol{\sigma}^{(1)} \cdot \mathbf{r}\, \alpha(1)\right) \left(\alpha(2)^\dagger \boldsymbol{\sigma}^{(2)} \cdot \mathbf{r}\, \alpha(2)\right) \\ &- \left(\alpha(1)^\dagger \boldsymbol{\sigma}^{(1)} \alpha(1)\right) \cdot \left(\alpha(2)^\dagger \boldsymbol{\sigma}^{(2)} \alpha(2)\right).\end{aligned}$$

Using

$$\alpha^\dagger \sigma_1 \alpha = \alpha^\dagger \sigma_2 \alpha = 0 \quad \text{and} \quad \alpha^\dagger \sigma_3 \alpha = 1,$$

this becomes

$$\psi_{11}^\dagger \widehat{S}_{12} \psi_{11} = \frac{3z^2}{r^2} - 1 = 3\cos^2\theta - 1,$$

or

$$\psi_{11}^\dagger \widehat{S}_{12} \psi_{11} = \sqrt{\frac{16\pi}{5}}\, Y_{20}(\theta, \phi).$$

This has $L = 2$, showing that the operator $\widehat{S}_{12}$ can transfer 2 units of angular momentum to the orbital motion of the particles.