1. Two particles with masses m_1 and m_2 collide with four-momenta

$$p_1 = (E_1, \mathbf{p}_1) \quad \text{and} \quad p_2 = (E_2, \mathbf{p}_2),$$

and produce two particles with masses m_3 and m_4 and four-momenta

$$p_3 = (E_3, \mathbf{p}_3) \quad \text{and} \quad p_4 = (E_4, \mathbf{p}_4).$$

The differential cross section for this process has the general form

$$\mathrm{d}\sigma = \frac{1}{F} \, |\mathcal{M}_{fi}|^2 \, \mathrm{d}Q,$$

where the invariant amplitude is $\mathcal{M}_{fi}$, the flux factor is

$$F = 4\,[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2},$$

and the Lorentz-invariant element of final phase space is

$$\mathrm{d}Q = (2\pi)^4 \, \delta^4(p_3 + p_4 - p_1 - p_2) \, \frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3} \, \frac{\mathrm{d}^3\mathbf{p}_4}{(2\pi)^3 2E_4} \, .$$

(a) Consider the collision in the centre-of-momentum frame, where

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}, \qquad \mathbf{p}_3 = -\mathbf{p}_4 = \mathbf{p}',$$

$$\mathbf{p} \cdot \mathbf{p}' = |\mathbf{p}||\mathbf{p}'| \cos\theta_{cm},$$

and

$$E_1 + E_2 = E_3 + E_4 = W.$$

Show that the flux factor can be written in this frame as

$$F = 4W|\mathbf{p}|.$$

By integrating over the redundant variables, $\mathbf{p}_4$ and $|\mathbf{p}_3|$, show that the element of invariant phase space can be written

$$\mathrm{d}Q = \frac{1}{(2\pi)^2} \, \frac{|\mathbf{p}'|}{4(E_3 + E_4)} \, \mathrm{d}\Omega_{cm},$$

where $\mathrm{d}\Omega_{cm} = \sin\theta_{cm} \, \mathrm{d}\theta_{cm} \, \mathrm{d}\phi$. Hence show that the angular differential cross section can be written

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{cm}} = \frac{1}{64\pi^2 W^2} \, \frac{|\mathbf{p}'|}{|\mathbf{p}|} \, |\mathcal{M}_{fi}|^2.$$

(b) Find expressions for the Lorentz invariants

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \quad \text{and} \quad t = (p_1 - p_3)^2 = (p_2 - p_4)^2,$$

in terms of the scattering variables in the centre-of-momentum frame. Use the expression for t to show that

$$\mathrm{d}t = 2|\mathbf{p}||\mathbf{p}'|\,\mathrm{d}(\cos\theta_{cm}),$$

and hence show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}|t|} = \frac{1}{64\pi W^2|\mathbf{p}|^2}\,|\mathcal{M}_{fi}|^2,$$

for a process where $|\mathcal{M}_{fi}|^2$ is independent of ϕ. Why is this differential cross section a Lorentz invariant?

(c) In the ultra-relativistic limit, $|\mathbf{p}| \gg m_1,\ m_2$, show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}|t|} = \frac{1}{16\pi s^2}\,|\mathcal{M}_{fi}|^2.$$

2. Consider the scattering of two spin-$\frac{1}{2}$ particles with charges q_1 and q_2 and masses m_1 and m_2. In the centre-of-momentum frame the particles have initial momenta

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p},$$

and final momenta

$$\mathbf{p}_1' = -\mathbf{p}_2' = \mathbf{p}',$$

where the scattering angle between $\mathbf{p}$ and $\mathbf{p}'$ is θ. The differential cross section for unpolarised particles may be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(8\pi W)^2}\,\frac{1}{4}\sum_{s_1 s_2 s_1' s_2'} |\mathcal{M}_{fi}|^2,$$

where W is the total energy and $\mathrm{d}\Omega$ is the element of solid angle in the centre-of-momentum frame.

(a) Show that the square of the transferred four-momentum is

$$(p_1' - p_1)^2 = -4\mathbf{p}^2 \sin^2\frac{\theta}{2}.$$

(b) Draw the Feynman diagram for the lowest-order contribution to this process and show that the corresponding invariant amplitude is

$$\mathcal{M}_{fi} = \mathrm{i}\,\bar{u}(\mathbf{p}_1', s_1')(-\mathrm{i}\,q_1\gamma^\mu)u(\mathbf{p}_1, s_1)\,\frac{-\mathrm{i}\,g_{\mu\nu}}{(p_1' - p_1)^2 + \mathrm{i}\epsilon}\,\bar{u}(\mathbf{p}_2', s_2')(-\mathrm{i}\,q_2\gamma^\nu)u(\mathbf{p}_2, s_2).$$

(c) Use the nonrelativistic approximation

$$\bar{u}(\mathbf{p}', s')\gamma^{\mu}u(\mathbf{p}, s) \simeq 2m\,\delta_{s's}\,\delta_{\mu 0}\,,$$

to show that the differential cross section for nonrelativistic scattering is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \simeq \frac{q_1^2 q_2^2}{64\pi^2}\left(\frac{m_1 m_2}{m_1+m_2}\right)^2 \frac{1}{|\mathbf{p}|^4 \sin^4\frac{\theta}{2}}\,.$$

(d) Now assume that particles 1 and 2 are identical fermions with mass m and charge q. Show that the non-relativistic differential cross section in this case is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q^4}{64\pi^2}\left(\frac{m}{2}\right)^2 \frac{1}{|\mathbf{p}|^4}\left[\frac{1}{\sin^4\frac{\theta}{2}} + \frac{1}{\cos^4\frac{\theta}{2}} - \frac{1}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right].$$

3. Consider the scattering of two spin-0 particles with charges q_1 and q_2 and masses m_1 and m_2. The kinematic variables and the cross section in the centre-of-momentum frame are defined in the same way as in the previous question.

(a) Draw the Feynman diagram for the lowest-order contribution to this process and show that the corresponding invariant amplitude is

$$\mathcal{M}_{fi} = \mathrm{i}\,(-\mathrm{i}\,q_1)(p_1^{\mu} + p_1'^{\mu})\,\frac{-\mathrm{i}\,g_{\mu\nu}}{(p_1' - p_1)^2 + i\epsilon}\,(-\mathrm{i}\,q_2)(p_2^{\nu} + p_2'^{\nu}).$$

(b) Show that the differential cross section for this process is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q_1^2 q_2^2}{64\pi^2 (E_1+E_2)^2}\,\frac{\left(E_1 E_2 + |\mathbf{p}|^2\cos^2\frac{\theta}{2}\right)^2}{|\mathbf{p}|^4 \sin^4\frac{\theta}{2}}\,.$$

(c) Show that in the nonrelativistic limit, $|\mathbf{p}| \ll m_1,\ m_2$, this cross section reduces to the usual Rutherford cross section.

(d) Find the form of the cross section in the limit where particle 2 is very heavy, $m_2 \gg |\mathbf{p}|,\ m_1$. Compare your result with the Mott cross section for scattering of a fermion from a heavy target, and explain why there is no suppression at backward angles in the spin-0 case.

4. [Postgraduate students only.]

(a) Use the Feynman rules for QED to write down the leading-order S-matrix element for the process $e^+e^- \to \gamma\gamma$.

(b) Show that the spin-averaged square of the amplitude for this process is

$$\frac{1}{4}\sum |M|^2 = 2\left[\frac{p_1 \cdot k_2}{p_1 \cdot k_1} + \frac{p_1 \cdot k_1}{p_1 \cdot k_2} + 1 - \left(1 - \frac{m^2(k_1 \cdot k_2)}{(p_1 \cdot k_1)(p_1 \cdot k_2)}\right)^2\right],$$

where p_1, p_2 are the electrons' momenta and k_1, k_2 are those of the photons. Hence obtain the unpolarised differential cross section for in the centre-of-mass frame. Integrate over angles to show that the total cross section is, in the ultra-relativistic limit where the electron velocity v is clise to 1,

$$\sigma \simeq \frac{\pi\alpha^2}{m^2}\frac{1-v^2}{4}\left[2\ln\left(\frac{2}{1-v}\right) - 2\right].$$

(c) Calculate the value for this total cross section at the energy of the Z^0 and compare it with the total cross section for $e^+e^- \to Z^0 \to$ hadrons. (The latter can be found in: Review of Particle Properties, Physics Letters B667 (2008).)

[*Hints*: You will need to evaluate sums over the spins of the electrons, which is best done by converting them into traces of Dirac matrices. The method is described in Aitchison and Hey 8.2.3, 8.2.4 and Gross 10.2, 10.3. Similar discussions can also be found in: Halzen and Martin 6.3, 6.4, Mandl and Shaw 8.2, and Itzykson and Zuber 5.2.1, with many useful identities collected in the appendices (Mandl and Shaw A.2, A.3, A.5 and Itzykson and Zuber A.2).

The evaluation of the polarisation sum is easiest if you use the 4-momentum of the electron, p_1, to define the polarisation vectors for the unphysical photons, so that

$$\epsilon_{(0)}(\mathbf{k}) = \frac{p_1}{m}, \qquad \epsilon_{(3)}(\mathbf{k}) = \frac{mk - (p_1 \cdot k)p_1/m}{p_1 \cdot k},$$

and the transverse polarisation vectors are orthogonal to both k and p_1. The evaluation of the traces then follows the calculation for Compton scattering, which is described in detail in Mandl and Shaw 8.6 and Itzykson and Zuber 5.2.1 (with appropriate substitutions of variables). Aitchison and Hey 8.6.3, Gross 10.5, and Halzen and Martin 6.14, 6.15 also discuss this process. For the polarisation basis suggested above, the term with the non-trivial polarisation sum can be expressed in the form

$$\sum_{s,s'=1}^{2}\left(\epsilon_{(s)}(\mathbf{k}_1) \cdot \epsilon_{(s')}(\mathbf{k}_2)\right)^2 = 1 + \left[1 - \frac{m^2(k_1 \cdot k_2)}{(p_1 \cdot k_1)(p_1 \cdot k_2)}\right]^2.$$

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