1. The *interaction picture* is often used to describe the time evolution of a quantum system whose Hamiltonian has the form

$$H = H_0 + H_I,$$

where H_0 is the Hamiltonian for freely moving particles and H_I describes the interactions between them. In this case, the time evolution of the system can be described using operators which evolve as in the Heisenberg picture, but governed by the free Hamiltonian,

$$A_I(t) \equiv \mathrm{e}^{\mathrm{i}H_0 t} A_S \, \mathrm{e}^{-\mathrm{i}H_0 t}.$$

Here A_S denotes the time-independent operator in the Schrödinger picture.

The states evolve as

$$|\psi(t)\rangle_I \equiv \mathrm{e}^{\mathrm{i}H_0 t} \, |\psi(t)\rangle_S,$$

where

$$|\psi(t)\rangle_S = \mathrm{e}^{-\mathrm{i}Ht} |\psi(0)\rangle_S,$$

is the Schrödinger-picture wave function, evolving according to the full Hamiltonian.

(a) Show that the equation of motion for the states in the interaction picture is

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle_I = H_I(t)\,|\psi(t)\rangle_I,$$

where

$$H_I(t) = \mathrm{e}^{\mathrm{i}H_0 t} H_I \, \mathrm{e}^{-\mathrm{i}H_0 t}.$$

Show also that the equation of motion for the operators in the interaction picture is

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,A_I(t) = \big[H_0, A_I(t)\big].$$

(b) The time evolution operator $U(t, t_i)$ in the interaction picture is defined by

$$|\psi(t)\rangle_I = U(t, t_0)\,|\psi(t_0)\rangle_I.$$

Show that the evolution operator is given by the infinite series

$$U(t, t_0) = 1 + (-\mathrm{i}) \int_{t_0}^{t} \mathrm{d}t_1 \, H_I(t_1) + (-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \, H_I(t_1) H_I(t_2) + \cdots .$$

2. The interaction Hamiltonian density responsible for the decay of a neutral pion to an electron-positron pair may be taken to be

$$\mathcal{H}(x) = g\,\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x)\,\phi(x)$$

where $\phi(x)$ and $\psi(x)$ are quantum field operators for the pion and the electron:

$$\phi(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2\omega(\mathbf{p})V}}\left(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger(\mathbf{p})\,\mathrm{e}^{+\mathrm{i}p\cdot x}\right),$$

and

$$\psi(x) = \sum_{\mathbf{p}s} \frac{1}{\sqrt{2E(\mathbf{p})V}}\left(b_s(\mathbf{p})\,u_s(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + d_s^\dagger(\mathbf{p})\,v_s(\mathbf{p})\mathrm{e}^{+\mathrm{i}p\cdot x}\right).$$

The energy of a pion is denoted here by

$$\omega(\mathbf{p}) = \sqrt{m_\pi^2 + \mathbf{p}^2},$$

and that of an electron by

$$E(\mathbf{p}) = \sqrt{m_e^2 + \mathbf{p}^2}.$$

(a) Show that the invariant amplitude for this decay is

$$\mathcal{M}_{fi} = g\bar{u}_s(\mathbf{k})\mathrm{i}\gamma_5 v_{s'}(\mathbf{k}'),$$

for a final electron with momentum $\mathbf{k}$ and spin s and a positron with momentum $\mathbf{k}'$ and spin s'.

(b) Show that the decay rate for a stationary pion to go to an electron and a positron is

$$W_{fi} = \frac{1}{2m_\pi}\,\frac{|\mathbf{k}|}{8\pi E(\mathbf{k})}\sum_{ss'}|\mathcal{M}_{fi}|^2,$$

where the momentum of the electron is

$$|\mathbf{k}| = \sqrt{\frac{m_\pi^2}{4} - m_e^2}.$$

(c) Given the spin sum

$$\sum_{ss'}|\bar{u}_s(\mathbf{k})\mathrm{i}\gamma_5 v_{s'}(\mathbf{k}')|^2 = 4(k\cdot k' + m_e^2),$$

show that the decay rate of the pion is

$$W_{fi} = \frac{g^2}{4\pi}\sqrt{\frac{m_\pi^2}{4} - m_e^2}.$$

3. A charged scalar boson (h^+ or h^-) can emit or absorb a neutral scalar boson (H). These processes can be described by the interaction Hamiltonian density

$$\mathcal{H}(x) = g\phi^\dagger(x)\phi(x)\Phi(x),$$

where the field operator for the charged boson is

$$\phi(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2\omega(\mathbf{p})V}} \big(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + c^\dagger(\mathbf{p})\,\mathrm{e}^{+\mathrm{i}p\cdot x}\big),$$

and that for the neutral boson is

$$\Phi(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2E(\mathbf{p})V}} \big(A(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + A^\dagger(\mathbf{p})\,\mathrm{e}^{+\mathrm{i}p\cdot x}\big).$$

(a) Draw the two Feynman diagrams which correspond to contributions of second order in g to the scattering amplitude of two identical h^+ bosons.

(b) Either by using the Feynman rules or (if you have time on your hands) by using the expansions of the field operators, evaluate the corresponding contributions to the invariant amplitude $\mathcal{M}_{fi}$. Hence show that the full invariant amplitude for this process is

$$\mathcal{M}_{fi} = g^2 \left[\frac{1}{(p_1 - p_3)^2 - m_H^2 + \mathrm{i}\epsilon} + \frac{1}{(p_1 - p_4)^2 - m_H^2 + \mathrm{i}\epsilon}\right],$$

where p_1 and p_2 are the momenta of the initial particles and p_3 and p_4 are those of the final.

(c) How would the amplitude be different for two h^- bosons? [This should require no extra calculation.]

(d) How would the amplitude be different for scattering of an h^+ and an h^- boson? [In this case you will need to draw and evaluate another Feynman diagram.]

(e) Find an approximate expression, $\mathcal{M}_{fi}^{NR}$, for the h^+h^- invariant amplitude which is valid when the particles' momenta are nonrelativistic.

(f) The nonrelativistic potential which gives the same scattering as $\mathcal{M}_{fi}^{NR}$ in the Born approximation is

$$V(\mathbf{r}) = \frac{1}{(2m_h)^2} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\, \mathcal{M}_{fi}^{NR}$$

where

$$\mathbf{q} = \mathbf{p_3} - \mathbf{p_1} = \mathbf{p_2} - \mathbf{p_4}.$$

Show that the nonrelativistic potential for h^+h^- scattering is

$$V(\mathbf{r}) = -\frac{g^2}{(2m_h)^2}\left[\frac{\mathrm{e}^{-m_H r}}{4\pi r} + \frac{1}{m_H^2 - (2m_h)^2}\,\delta^3(\mathbf{r})\right].$$

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