1. A Hermitian Klein-Gordon field in one dimension is described by the quantum field operator

$$\phi(t,z) = \sum_n \frac{1}{\sqrt{2E_n L}}\big(a_n\,\mathrm{e}^{-\mathrm{i}(E_n t - k_n z)} + a_n^\dagger\,\mathrm{e}^{+\mathrm{i}(E_n t - k_n z)}\big).$$

The field is defined in a box of length L with periodic boundary conditions and so

$$k_n = \frac{2\pi n}{L}.$$

The operators a_n and $a_n^\dagger$ can be assumed to satisfy the commutation relations

$$\big[a_n, a_m^\dagger\big] = \delta_{nm}, \qquad \big[a_n, a_m\big] = \big[a_n^\dagger, a_m^\dagger\big] = 0.$$

The Hamiltonian for the field is

$$H = \int_0^L \mathrm{d}z\,\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + m^2\phi^2\right],$$

where

$$\frac{\partial\phi}{\partial t} = -\mathrm{i}\sum_n \sqrt{\frac{E_n}{2L}}\big(a_n\,\mathrm{e}^{-\mathrm{i}(E_n t - k_n z)} - a_n^\dagger\,\mathrm{e}^{+\mathrm{i}(E_n t - k_n z)}\big).$$

(a) Show that the Hamitonian can be written in terms of the operators a_n and $a_n^\dagger$ as

$$H = \sum_n E_n\left(a_n^\dagger a_n + \tfrac{1}{2}\right),$$

provided that E_n is taken to be

$$E_n = \sqrt{m^2 + k_n^2}.$$

(b) Evaluate the commutators

$$\big[H, a_n\big] \quad \text{and} \quad \big[H, a_n^\dagger\big].$$

Hence show that a_n decreases the energy by E_n and $a_n^\dagger$ increases the energy by E_n.

(c) Evaluate the commutator

$$\big[H, \phi(t,z)\big],$$

and hence show that $\phi(t,z)$ satisfies the Heisenberg equation of motion.

$$\big[H, \phi(t,z)\big] = -\mathrm{i}\,\frac{\partial\phi}{\partial t}.$$

(d) The momentum operator is defined by

$$P = -\int_0^L \mathrm{d}z \, \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial z}.$$

Show that P can be written terms of the operators a_n and $a_n^\dagger$ as

$$P = \sum_n k_n a_n^\dagger a_n.$$

(e) Evaluate the commutators

$$\big[P, a_n\big] \quad \text{and} \quad \big[P, a_n^\dagger\big].$$

Discuss why the quanta destroyed and created by a_n and $a_n^\dagger$ can be interpreted as relativistic particles with mass m.

(f) Evaluate the commutator

$$\big[P, \phi(t,z)\big],$$

and hence show that P can be regarded as the generator of translations in space.

(g) Evaluate the equal-time commutators

$$\left[\frac{\partial\phi}{\partial t}(t,z), \phi(t,z')\right] \quad \text{and} \quad \big[\phi(t,z), \phi(t,z')\big].$$

2. A complex Klein-Gordon field is described by the quantum field operator

$$\phi(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2E(\mathbf{p})V}}\big(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x} + c^\dagger(\mathbf{p})\,\mathrm{e}^{+\mathrm{i}p\cdot x}\big),$$

where

$$p\cdot x = E(\mathbf{p})\,t - \mathbf{p}\cdot\mathbf{x} \quad \text{and} \quad E(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}.$$

Assume that all the commutators involving the creation and destruction operators vanish except for

$$\big[a(\mathbf{p}), a^\dagger(\mathbf{p}')\big] = \delta_{\mathbf{p}\mathbf{p}'} \quad \text{and} \quad \big[c(\mathbf{p}), c^\dagger(\mathbf{p}')\big] = \delta_{\mathbf{p}\mathbf{p}'}.$$

The Hamiltonian for the field is (ignoring any infinite constant term)

$$H = \sum_{\mathbf{p}} E(\mathbf{p})\,\big(a^\dagger(\mathbf{p})\,a(\mathbf{p}) + c^\dagger(\mathbf{p})\,c(\mathbf{p})\big)\,.$$

(a) Evaluate the commutators

$$\big[H, a(\mathbf{p})\big] \quad \text{and} \quad \big[H, c(\mathbf{p})\big].$$

Hence show that the operators $a(\mathbf{p})$ and $c(\mathbf{p})$ decrease the energy by $E(\mathbf{p})$ and that the operators $a^\dagger(\mathbf{p})$ and $c^\dagger(\mathbf{p})$ increase the energy by $E(\mathbf{p})$.

(b) Show that $\phi(x)$ satisfies the Heisenberg equation of motion

$$-\mathrm{i}\,\frac{\partial\phi}{\partial t} = \big[H, \phi\big].$$

(c) The charge operator Q is given by

$$Q = \int \mathrm{d}^3\mathbf{x}\, \rho(x),$$

where $\rho(x)$ is the Klein-Gordon density

$$\rho(x) = \mathrm{i}\left[\phi^\dagger(x)\,\frac{\partial\phi(x)}{\partial t} - \frac{\partial\phi^\dagger(x)}{\partial t}\,\phi(x)\right].$$

Evaluate the commutators

$$\big[Q, a(\mathbf{p})\big] \quad \text{and} \quad \big[Q, c(\mathbf{p})\big].$$

Hence show that the operators $a(\mathbf{p})$ and $c^\dagger(\mathbf{p})$ decrease the charge by one unit and that the operators $a^\dagger(\mathbf{p})$ and $c(\mathbf{p})$ increase the charge by one unit.

3. A Dirac field is described by the quantum field operator

$$\psi(x) = \sum_{\mathbf{p},s} \frac{1}{\sqrt{2E(\mathbf{p})V}}\big(b_s(\mathbf{p})\,u_s(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\cdot x} + d_s^\dagger(\mathbf{p})\,v_s(\mathbf{p})\mathrm{e}^{+\mathrm{i}p\cdot x}\big),$$

where

$$p\cdot x = E(\mathbf{p})\,t - \mathbf{p}\cdot\mathbf{x} \quad \text{and} \quad E(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}.$$

The free-particle spinors, $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$, are covariantly normalised and orthogonal:

$$u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) = 2E(\mathbf{p})\delta_{ss'}, \qquad v_s^\dagger(\mathbf{p})v_{s'}(\mathbf{p}) = 2E(\mathbf{p})\delta_{ss'},$$

and

$$u_s^\dagger(\mathbf{p})v_{-s'}(-\mathbf{p}) = 0.$$

Assume that all the anticommutators involving the creation and destruction operators vanish except for

$$\big\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\big\} = \delta_{ss'}\delta_{\mathbf{p}\mathbf{p}'} \quad \text{and} \quad \big\{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\big\} = \delta_{ss'}\delta_{\mathbf{p}\mathbf{p}'}.$$

(a) The Hamiltonian for the field is

$$H = \int \mathrm{d}^3\mathbf{x}\, \psi^\dagger\big(-\mathrm{i}\,\boldsymbol{\alpha}\cdot\nabla + \beta m\big)\psi.$$

Show that this can be rewritten (up to an infinite constant) as

$$H = \sum_{\mathbf{p},s} E(\mathbf{p})\,\big(b_s^\dagger(\mathbf{p})\,b_s(\mathbf{p}) + d_s^\dagger(\mathbf{p})\,d_s(\mathbf{p})\big)\,.$$

(b) Evaluate the commutators

$$\big[H, b_s(\mathbf{p})\big] \quad \text{and} \quad \big[H, d_s(\mathbf{p})\big].$$

Hence show that the operators $b_s(\mathbf{p})$ and $d_s(\mathbf{p})$ decrease the energy by $E(\mathbf{p})$ and that the operators $b_s^\dagger(\mathbf{p})$ and $d_s^\dagger(\mathbf{p})$ increase the energy by $E(\mathbf{p})$.

(c) Show that $\psi(x)$ satisfies the Heisenberg equation of motion

$$-\mathrm{i}\,\frac{\partial\psi}{\partial t} = \big[H, \psi\big].$$

(d) The charge operator Q is given by

$$Q = \int \mathrm{d}^3\mathbf{x}\,\rho(x),$$

where $\rho(x)$ is the Dirac density

$$\rho(x) = \psi^\dagger(x)\psi(x).$$

Evaluate the commutators

$$\big[Q, b_s(\mathbf{p})\big] \quad \text{and} \quad \big[Q, d_s(\mathbf{p})\big].$$

Hence show that the operators $b_s(\mathbf{p})$ and $d_s^\dagger(\mathbf{p})$ decrease the charge by one unit and that the operators $b_s^\dagger(\mathbf{p})$ and $d_s(\mathbf{p})$ increase the charge by one unit.

4. The Feynman propagator for a complex Klein-Gordon field is defined by

$$G_F(x) = -\mathrm{i}\,\langle 0|\mathrm{T}[\phi(x)\phi^\dagger(0)]|0\rangle,$$

where the time-ordered product is defined as

$$\mathrm{T}[\phi(x)\phi^\dagger(0)] = \theta(t)\phi(x)\phi^\dagger(0) + \theta(-t)\phi^\dagger(0)\phi(x).$$

Here $\theta(t)$ is the step function

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases},$$

whose derivative is the Dirac δ-function

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \delta(t).$$

(a) By expanding the field operators as in question 2, show that $G_F(x)$ can be written

$$G_F(x) = -\mathrm{i}\sum_{\mathbf{p}}\left[\theta(t)\,\frac{\mathrm{e}^{-\mathrm{i}(Et-\mathbf{p}\cdot\mathbf{x})}}{2E(\mathbf{p})V} + \theta(-t)\,\frac{\mathrm{e}^{\mathrm{i}(Et-\mathbf{p}\cdot\mathbf{x})}}{2E(\mathbf{p})V}\right],$$

where $E(\mathbf{p}) = \sqrt{m^2+\mathbf{p}^2}$. Give a physical interpretation for each of the two sets of terms in this expansion.

(b) Show that $G_F(x)$ satisfies the inhomogeneous Klein-Gordon equation

$$\left(\partial_\mu\partial^\mu + m^2\right) G_F(x) = -\delta^4(x).$$

[You can do this either by diferentiating the original expression for $G_F(x)$ and using the equal-time commutation relations,

$$\left[\phi(0,\mathbf{x}), \phi^\dagger(0,\mathbf{x}')\right] = 0 \quad \text{and} \quad \left[\frac{\partial\phi}{\partial t}(0,\mathbf{x}), \phi^\dagger(0,\mathbf{x}')\right] = -\mathrm{i}\,\delta^3(\mathbf{x}-\mathbf{x}'),$$

or by differentiating the expansion in part (a).]

(c) The result of part (b) shows that $G_F(x)$ is a Green's function for the Klein-Gordon equation. By taking the Fourier transform of this equation, show that the momentum-space Green's function,

$$\widetilde{G}_F(k) = \int \mathrm{d}^4x\, G_F(x)\, \mathrm{e}^{\mathrm{i}k\cdot x},$$

satisfies

$$(k^2 - m^2)\, \widetilde{G}_F(k) = 1.$$

Hence show that $G_F(x)$ can be written

$$G_F(x) = \frac{1}{2\pi V}\int \mathrm{d}k^0 \sum_{\mathbf{k}} \frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{k^2 - m^2}.$$

[This expression ignores the fact that we need to impose the correct boundary conditions on the Green's function by specifying how the singularities at $k^2 = m^2$ should be avoided.]

(d) [For students who have taken a course on complex variables.] The Feynman prescription for avoiding the singularities of $\widetilde{G}_F(k)$ is to add $+\mathrm{i}\epsilon$ to the denominator:

$$\widetilde{G}_F(k) = \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon}.$$

Choose suitable contours in the complex k^0 plane for the cases $t > 0$ and $t < 0$. By integrating the result of part (c) over k^0 using these contours, show that the Feynman prescription does lead to the same expression for $G_F(x)$ as you found in part (a).

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