1. The four-component spinor wave function $\psi(t, \mathbf{x})$ of a free spin-$\frac{1}{2}$ particle of mass m is described by the Dirac equation

$$\mathrm{i}\,\frac{\partial\psi}{\partial t} = H\psi,$$

where the Dirac Hamiltonian is

$$H = -\mathrm{i}\,\boldsymbol{\alpha}\cdot\nabla + \beta m.$$

In the Pauli-Dirac representation, the 4×4 matrices are

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

(a) Show that these matrices satisfy the anticommutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \qquad \{\alpha_i, \beta\} = 0,$$

and hence that ψ satisfies the Klein-Gordon equation.

(b) Show that

$$\rho(x) = \psi^\dagger\psi \quad \text{and} \quad \mathbf{j}(x) = \psi^\dagger\boldsymbol{\alpha}\psi$$

satisfy the equation of continuity

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{j}.$$

(c) Show that the orbital angular momentum operator $\mathbf{L} = \mathbf{x}\times\mathbf{p}$ (where $\mathbf{p} = -\mathrm{i}\nabla$) is not a constant of motion; show in particular that

$$[H, \mathbf{L}] = -\mathrm{i}\,\boldsymbol{\alpha}\times\mathbf{P}.$$

(d) Consider the operator

$$\mathbf{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

What are the eigenvalues of $\mathbf{S}^2$ and S_i?

(e) Show that

$$[\alpha_i, S_j] = \mathrm{i}\,\epsilon_{ijk}\,\alpha_k \quad \text{and} \quad [\beta, S_j] = 0.$$

Hence show that $\mathbf{S}$ is not a constant of motion, but that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is. What properties do these operators describe?

(f) Show that $\mathbf{S}\cdot\mathbf{P}$ is also a constant of motion. What property does it describe?

2. The wave function of a free spin-$\frac{1}{2}$ particle satisfies the time-independent Dirac equation

$$E\psi = [-\mathrm{i}\,\boldsymbol{\alpha}\cdot\nabla + \beta m]\psi.$$

(a) Working in the Pauli-Dirac representation, show that

$$\psi^{(+)}_{\mathbf{p},s}(\mathbf{x}) = u_s(\mathbf{p})\,\frac{\mathrm{e}^{+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2E(\mathbf{p})V}},$$

where

$$u_s(\mathbf{p}) = \sqrt{E(\mathbf{p})+m}\begin{pmatrix} \chi_{+s} \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E(\mathbf{p})+m}\,\chi_{+s}\end{pmatrix},$$

is a solution to this equation and describes a particle with momentum $\mathbf{p}$ and energy $E(\mathbf{p}) = +\sqrt{\mathbf{p}^2+m^2}$. Show that the solution is normalised in a box of volume V. Show that if $\mathbf{p} = (0,0,p)$, the particle has definite spin $S_3 = s$. [Here χ_s, with $s = \pm\frac{1}{2}$, denotes a normalised eigenvector of σ_3.]

(b) Show that

$$\psi^{(-)}_{\mathbf{p},s}(\mathbf{x}) = v_s(\mathbf{p})\,\frac{\mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2E(\mathbf{p})V}},$$

where

$$v_s(\mathbf{p}) = \sqrt{E(\mathbf{p})+m}\begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E(\mathbf{p})+m}\,\chi_{-s} \\ \chi_{-s}\end{pmatrix},$$

is also a solution to the Dirac equation and describes a particle with momentum $-\mathbf{p}$ and negative energy $-E(\mathbf{p})$.

(c) Evaluate the density, $\rho = \psi^\dagger\psi$, the current density, $\mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi$, and the scalar density $\rho_s = \overline{\psi}\psi = \psi^\dagger\beta\psi$, for the positive- and negative-energy wave functions you found in parts (a) and (b).

3. The Weyl (pronounced "vile") representation for the Dirac matrices is

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma}\end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},$$

in terms of the Pauli matrices $\boldsymbol{\sigma}$.

(a) Confirm that these matrices satisfy the anticommutation relations given above in question 1.

(b) Working in the Weyl representation, write the Dirac wave function as

$$\psi(x) = N\mathrm{e}^{-\mathrm{i}p\cdot x}\begin{pmatrix} \zeta \\ \eta\end{pmatrix},$$

and use this to convert the Dirac equation,

$$i\frac{\partial\psi}{\partial t} = -\mathrm{i}\,\boldsymbol{\alpha}\cdot\nabla\psi + \beta m\psi,$$

into a pair of coupled matrix equations for ζ and η.

(c) Show that for a massless particle ($m = 0$) these equations decouple to become

$$E\zeta = +\boldsymbol{\sigma} \cdot \mathbf{p}\zeta,$$

and

$$E\eta = -\boldsymbol{\sigma} \cdot \mathbf{p}\eta.$$

(d) Find the energy eigenvalues for the equation for ζ and show that they agree with the expected energy-momentum relation for a massless particle.

(e) The *helicity* of a spin-$\frac{1}{2}$ particle is defined by the operator

$$\mathbf{S} \cdot \hat{\mathbf{p}} = \frac{1}{2|\mathbf{p}|} \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix}.$$

Show that the positive-energy solution of the equation for ζ describes a particle with definite helicity, $+\frac{1}{2}$.

(f) Show that the negative-energy solution of the equation for ζ describes an antiparticle with definite helicity, $-\frac{1}{2}$. [Remember that you need to replace $E(\mathbf{p}) \to -E(\mathbf{p})$, $\mathbf{p} \to -\mathbf{p}$, and $s \to -s$ to get the wave function of an antiparticle.]

4. The manifestly covariant form of the Dirac equation is

$$[i\gamma^\mu \partial_\mu - m]\psi(x) = 0.$$

(a) Show that $\psi(x)$ satisfies the Klein-Gordon equation if

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.$$

(b) Consider Dirac wave functions with the forms

$$\psi^{(+)}(x) = u_s(\mathbf{p}) \, \frac{\mathrm{e}^{-\mathrm{i}p\cdot x}}{\sqrt{2E_{\mathbf{p}}V}} \quad \text{and} \quad \psi^{(-)}(x) = v_s(\mathbf{p}) \, \frac{\mathrm{e}^{+\mathrm{i}p\cdot x}}{\sqrt{2E_{\mathbf{p}}V}}.$$

Show that these are solutions to the Dirac equation if the spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ satisfy the matrix equations

$$[\gamma^\mu p_\mu - m]u_s(\mathbf{p}) = 0 \quad \text{and} \quad [\gamma^\mu p_\mu + m]v_s(\mathbf{p}) = 0.$$

(c) Write down the covariant form of the Dirac equation if the particle has charge q and interacts with an electromagnetic field described by the 4-vector potential $A^\mu(x)$.

(d) The Dirac conjugate of ψ is defined by

$$\overline{\psi} = \psi^\dagger \gamma_0.$$

Use the property

$$\gamma^{\mu\dagger} = \gamma_0 \gamma^\mu \gamma_0$$

to show that $\overline{\psi}$ satisfies the Dirac equation

$$-\mathrm{i}\,(\partial_\mu \overline{\psi})\gamma^\mu - m\overline{\psi} = 0.$$

(e) Show that the current density

$$j^\mu = \overline{\psi}\gamma^\mu \psi$$

satisfies the equation of continuity.

5. The wave function, $\psi(\mathbf{r})$, of a relativistic spin-half particle with energy E in a potential $V(r)$ satisfies the Dirac equation

$$[-\mathrm{i}\,\boldsymbol{\alpha}\cdot\nabla + \beta m + V(r)]\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

Working in the Pauli-Dirac representation (see question 1), consider a solution with $j = \frac{1}{2}$ and even parity,

$$\psi(\mathbf{r}) = \begin{pmatrix} f(r)\,\chi \\ \mathrm{i}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\,g(r)\,\chi \end{pmatrix},$$

where χ denotes a constant Pauli spinor.

(a) Show that the radial functions $f(r)$ and $g(r)$ satisfy the coupled differential equations

$$\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{2g(r)}{r} + [m + V(r) - E]f(r) = 0, \qquad (1)$$

$$\frac{\mathrm{d}f}{\mathrm{d}r} + [m - V(r) + E]g(r) = 0 \qquad (2).$$

You may assume that

$$(\boldsymbol{\sigma}\cdot\nabla)(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})g(r) = \frac{2g(r)}{r} + \frac{\mathrm{d}g}{\mathrm{d}r}.$$

(b) Assume that the potential $V(r)$ has the form

$$V(r) = \begin{cases} 0 & \text{if } r < R \\ V_0 & \text{if } r > R \end{cases}.$$

Obtain a second-order differential equation satisfied by $f(r)$ in the region $r > R$ and find its general solution.

(c) Show that a particle with energy E cannot be confined if the potential barrier is too low, $V_0 < E - m$, or too high, $V_0 > E + m$.

(d) Suggest a reason why a high barrier is not able to confine the particle.

6. The Dirac Hydrogen atom. [This is a harder question and should be tackled only when you are happy with question 5 and the Klein-Gordon Hydrogen atom.] Start from equations (1) and (2) in question 5(a) for the $s_{\frac{1}{2}}$ levels. Act on equation (1) with $\frac{\mathrm{d}}{\mathrm{d}r}$ and on (2) with $\frac{\mathrm{d}}{\mathrm{d}r} + \frac{2}{r}$. You should end up with two nearly decoupled equations,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{2}{r^2} - m^2 + (E-V)^2\right] g + \frac{\mathrm{d}V}{\mathrm{d}r} f = 0,$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - m^2 + (E-V)^2\right] f - \frac{\mathrm{d}V}{\mathrm{d}r} g = 0.$$

Now consider the case of the Coulomb potential,

$$V(r) = -\frac{Z\alpha}{r}.$$

The two equations above are identical except for the $1/r^2$ terms, which can be written in matrix form as

$$-\frac{1}{r^2}\begin{pmatrix} -(Z\alpha)^2 & Z\alpha \\ -Z\alpha & 2-(Z\alpha)^2 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \equiv -\frac{1}{r^2} A \begin{pmatrix} f \\ g \end{pmatrix}.$$

Find the eigenvalues of the matrix A and show that the positive one can be written in the form $l'(l'+1)$ where

$$l' = \sqrt{1-(Z\alpha)^2}.$$

[You can discard the negative eigenvalue since it does not lead to solutions of the differential equation which satisfy the appropriate boundary conditions.]

Call the corresponding eigenvector of A $\mathbf{e}_+$ (you do not need to find its detailed form) and show that

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = f(r)\,\mathbf{e}_+$$

can be a solution to the equations above provided that $f(r)$ satisfies the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l'(l'+1)}{r^2} - m^2 + E^2 + 2E\frac{Z\alpha}{r}\right] f = 0.$$

This now has exactly the same form as the Schrödinger equation for the Hydrogen atom, but with different constants. Proceed as we did in the Klein-Gordon case and use the known energy levels for the Schrödinger atom to write down the levels for the Dirac one. In the limit $Z\alpha \ll 1$ you should find that the $s_{\frac{1}{2}}$ levels are

$$E \simeq \left[1 - \frac{(Z\alpha)^2}{2n^2} - \frac{Z\alpha)^4}{2n^4}\left(\frac{n}{2} - \frac{3}{4}\right)\right].$$

Comment on what happens to the energies for large $Z\alpha$ and suggest an expanation.

Mike Birse (January 2010)