1. The wave function $\phi(\mathbf{x}, t)$ satisfies the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = 0.$$

Show that

$$\rho = \mathrm{i}\left[\phi^* \frac{\partial \phi}{\partial t} - \left(\frac{\partial \phi^*}{\partial t}\right)\phi\right] \quad \text{and} \quad \mathbf{j} = -\mathrm{i}\left[\phi^* \nabla \phi - (\nabla \phi^*)\phi\right]$$

obey the equation of continuity

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}.$$

Evaluate ρ and $\mathbf{j}$ for the wave functions:

$$\phi^{(+)}(t, \mathbf{x}) = \frac{1}{\sqrt{2EV}}\,\mathrm{e}^{-\mathrm{i}(Et - \mathbf{p}\cdot\mathbf{x})} \quad \text{and} \quad \phi^{(-)}(t, \mathbf{x}) = \frac{1}{\sqrt{2EV}}\,\mathrm{e}^{+\mathrm{i}(Et - \mathbf{p}\cdot\mathbf{x})}.$$

2. The Klein-Gordon equation for a particle of charge q in the presence of an electromagnetic field is

$$[(\mathrm{i}\partial^\mu - qA^\mu)(\mathrm{i}\partial_\mu - qA_\mu) - m^2]\phi(x) = 0,$$

where $A^\mu = (A^0, \mathbf{A})$ is the 4-vector potential specifying the field. Show that the 4-vector current density

$$j^\mu(x) = \phi^*(\mathrm{i}\partial^\mu - qA^\mu)\phi - \phi(\mathrm{i}\partial^\mu + qA^\mu)\phi^*,$$

obeys the equation of continuity

$$\partial_\mu j^\mu = 0.$$

Find expressions for the density $\rho = j^0$ and the current density $\mathbf{j}$ for the case where there is only a potential energy, $V = qA^0$ and $\mathbf{A} = 0$.

3. A particle moving in a potential V is described by the Klein-Gordon equation

$$[-(E - V)^2 - \nabla^2 + m^2]\psi = 0.$$

Consider the limit where the potential is weak and the energy is low,

$$|V| \ll m \quad \text{and} \quad |\mathcal{E}| \ll m,$$

where $\mathcal{E} = E - m$, Show that in this limit the KG equation can be approximated by the Schrödinger equation

$$[-\nabla^2 + 2mV]\psi = 2m\mathcal{E}\psi.$$

4. A particle of energy E, mass m and charge q is incident on an electrostatic barrier where its potential energy increases abruptly from $V = 0$ for $z < 0$ to $V = V_0$ for $z > 0$. The particle is described by the Klein-Gordon equation

$$[-(E - V)^2 - \nabla^2 + m^2]\psi = 0.$$

Its wave function has the form

$$\psi(z) = A\,\mathrm{e}^{\mathrm{i}pz} + B\,\mathrm{e}^{-\mathrm{i}pz} \quad \text{for} \quad x < 0,$$

and

$$\psi(z) = C\,\mathrm{e}^{\mathrm{i}p'z} \quad \text{for} \quad z > 0.$$

(a) Show that

$$p^2 = E^2 - m^2 \quad \text{and} \quad p'^2 = (E - V_0)^2 - m^2.$$

(b) Show that

$$B = \frac{p - p'}{p + p'}A \quad \text{and} \quad C = \frac{2p}{p + p'}A.$$

(c) Evaluate the current densities for each of the three parts of the wave (incident, reflected and transmitted),

$$\mathbf{j} = -\mathrm{i}\,[\psi^*\nabla\psi - (\nabla\psi^*)\psi],$$

assuming that p' is real. Check that the current is conserved.

(d) First, consider a weak potential with $V_0 < E$. Show that when $V_0 < E - m$ a wave propagates into the region $z > 0$ with group velocity

$$v_g = \frac{p'}{E - V_0}.$$

Show that in this situation the current associated with the reflected wave is smaller in magnitude than the current associated with the incident wave, and that the currents associated with the incident and transmitted waves have the same signs.

(e) Now consider a strong potential with $V_0 > E$. Show that when $V_0 > E + m$ a wave propagates into the region $z > 0$ with group velocity

$$v_g = -\frac{p'}{V_0 - E}.$$

Show that in this situation the current associated with the reflected wave is greater in magnitude than the current associated with the incident wave, and that the currents associated with the incident and transmitted waves have opposite signs.

(f) Suggest a possible interpretation of this paradoxical result. Use this interpretation to explain the unstable states we found in lectures for the Klein-Gordon Hydrogen atom with $Z > 69$.

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