The problems on this sheet are intended to familiarise you with the use of natural units and to refresh your knowledge of relevant ideas from relativity and quantum mechanics.

Questions 1 to 4 are on ideas that we shall be using right from the start of the course: natural units and relativistic notation. You should try them as soon as possible. If you find that your knowledge of special relativity is rusty, you should look at: W. Rindler, Introduction to special relativity, 2nd edition (Oxford, 1991), Chapters IV–VI. If you feel that you need to refresh your knowledge of nonrelativistic quantum mechanics, you should also look at: S. Gasiorowicz, Quantum physics (Wiley, 1974), Chapters 3 and 4.

Question 5 (Pauli matrices) and 6 (the vector potential) are needed for the Dirac equation, which we shall meet at the end of week 2. If you are unsure about these you should study Gasiorowicz, Chapters 13 and 14. Question 7 (creation and annihilation operators) should be done before we start setting up quantum field theory in week 4, and is covered in Gasiorowicz, Chapter 7. Questions 8 (change of variables and the δ-function) and 9 (Schrödinger and Heisenberg pictures) are needed before we start using field theory in week 6. The latter is also covered in Gasiorowicz, Chapter 7. You should work through these chapters if the ideas in them are unfamiliar.

1. Show that the energy levels of the Hydrogen atom can be written in natural units as
$$E_n = -\frac{\alpha^2 m}{2n^2},$$
where the fine structure constant is $\alpha = 1/137$ and the mass of the electron is $m = 0.51$ MeV. Check that this gives the usual numerical value for the ground state energy in eV.

   Show that the Bohr radius of the Hydrogen atom can be written in natural units as
$$a = \frac{1}{\alpha m}.$$
Evaluate this in MeV^{-1}. Convert this to more familiar length-based units with the help of the very useful constant $\hbar c \simeq 200$ MeV fm.

2. Rewrite the relativistic energy-momentum relation as a dispersion relation connecting angular frequency ω and wave number $\mathbf{k}$ in natural units. Hence find the corresponding group and phase velocities, and comment on your results.

3. Two particles of mass m_1 and m_2 collide with 4-momenta

$$p_1 = (E_1, \mathbf{p}_1) \quad \text{and} \quad p_2 = (E_2, \mathbf{p}_2)$$

and produce two particles of mass m_3 and m_4 with 4-momenta

$$p_3 = (E_3, \mathbf{p}_3) \quad \text{and} \quad p_4 = (E_4, \mathbf{p}_4).$$

The kinematics of this scattering process may be described in terms of the Lorentz invariant quantities (Mandelstam variables)

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2 \quad \text{and} \quad u = (p_1 - p_4)^2.$$

Using conservation of 4-momentum, show that these satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2.$$

In the center-of momentum (cm) frame the incoming momenta are related by $\mathbf{p}_2 = -\mathbf{p}_1$. The cm scattering angle θ is defined by $\mathbf{p}_1 \cdot \mathbf{p}_3 = |\mathbf{p}_1||\mathbf{p}_3| \cos\theta$. Show that in this frame we can write

$$s = (E_1 + E_2)^2, \qquad t = m_1^2 + m_3^2 - 2E_1 E_3 + 2|\mathbf{p}_1||\mathbf{p}_3| \cos\theta.$$

4. (a) A frame S' moving along the x-axis with velocity v relative a frame S. The space-time coordinates $(t', \mathbf{x}')$ of an event in S' are related to its coordinates $(t, \mathbf{x})$ in S by the Lorentz transformation

$$\begin{aligned} t' &= \frac{1}{\sqrt{1 - v^2}} (t - vx) \\ x' &= \frac{1}{\sqrt{1 - v^2}} (x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

Write down the inverse transformation which relates $(t, \mathbf{x})$ to $(t', \mathbf{x}')$. Use this and the chain rule for partial differentiation to show that the gradient of a scalar field $\phi(x)$,

$$\partial_\mu \phi = \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right),$$

transforms as a covariant 4-vector.

(b) Evaluate the derivatives

$$\partial^\mu e^{-\mathrm{i}p \cdot x} \qquad \text{and} \qquad \partial^\mu \partial_\mu e^{-\mathrm{i}p \cdot x},$$

where $p \cdot x = p_\mu x^\mu = p^\mu x_\mu$. How does each of these objects transform under Lorentz transformations?

5. The Pauli spin matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Evaluate their commutators $[\sigma_i, \sigma_j]$ and their anticommutators $\{\sigma_i, \sigma_j\}$.

Show that the commutators can be expressed in the form

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k,$$

where ϵ_{ijk} is the antisymmetric symbol ($\epsilon_{ijk} = +1$ if ijk is a cyclic permutation of 123, -1 if it is anticyclic, and 0 otherwise). Show also that the anticommutators can be expressed in the form

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1},$$

where $\mathbf{1}$ denotes the 2×2 unit matrix.

Hence show that

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + \mathrm{i}\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}),$$

where $\mathbf{a}$ and $\mathbf{b}$ are 3-vectors.

6. A magnetic field can always be expressed in terms of a vector potential $\mathbf{A}$ as $\mathbf{B} = \nabla \times \mathbf{A}$, since this guarantees that Gauss's law is satisfied.

Verify that the potential

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B}_0 \times \mathbf{x}$$

corresponds to a uniform magnetic field $\mathbf{B}_0$.

The Hamiltonian for a nonrelativistic particle of charge q moving in the presence of this field is

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m},$$

where $\mathbf{p} = -i\nabla$ is the momentum operator. Show that this Hamiltonian can be written in the form

$$H = \frac{\mathbf{p}^2}{2m} - \frac{q}{2m}\mathbf{B}_0 \cdot \mathbf{L} + \frac{q^2}{8m}\left(|\mathbf{x}|^2|\mathbf{B}_0|^2 - (\mathbf{x} \cdot \mathbf{B}_0)^2\right),$$

where $\mathbf{L} = -i\mathbf{x} \times \nabla$ is the orbital angular momentum operator. Give a physical interpretation for the second term in H.

7. What is the commutator of the one-dimensional position and momentum operators, x and p, in natural units? These operators can be written in terms of creation and annihilation operators, $a^\dagger$ and a, as

$$x = \frac{1}{\sqrt{2\omega}}\,(a + a^\dagger) \quad \text{and} \quad p = -\mathrm{i}\sqrt{\frac{\omega}{2}}\,(a - a^\dagger).$$

Find the commutator of a and $a^\dagger$.

A quantum mechanical harmonic oscillator has the Hamiltonian

$$H = \frac{1}{2}\,p^2 + \frac{1}{2}\,\omega^2 x^2.$$

Express this in terms of a and $a^\dagger$ and hence find the energy eigenvalues and eigenstates of H.

8. The Dirac δ-function of a function $f(x)$ is equivalent to a sum of δ-functions of x at the zeros of $f(x)$:

$$\delta\Big(f(x)\Big) = \sum_i \frac{1}{|f'(x_i)|}\,\delta(x - x_i),$$

where $f(x_i) = 0$.

When integrating over the invariant phase space of a relativistic particle, we shall meet integrals of the form

$$\int \mathrm{d}k^0\,\mathrm{d}^3\mathbf{k}\,\delta\Big((k^0)^2 - \mathbf{k}^2 - m^2\Big)\,G(k^0, \mathbf{k}),$$

where G is some function of the 4-momentum of the particle and the δ-function enforces the condition $k^2 = m^2$. Using the above property of the δ-function, show that this integral may be rewritten in the form

$$\int \frac{\mathrm{d}^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}}\left[G\left(\sqrt{\mathbf{k}^2 + m^2}, \mathbf{k}\right) + G\left(-\sqrt{\mathbf{k}^2 + m^2}, \mathbf{k}\right)\right].$$

9. The time evolution of a quantum system is governed by its Hamiltonian. Outline how this idea is implemented in (a) the Schrödinger picture and in (b) the Heisenberg picture.

Mike Birse (December 2009)