PHYS30201 Mathematical Fundamentals of Quantum Mechanics **Examples 4**

1. A particle has spin $s = \frac{3}{2}$ and orbital angular momentum $l = 1$. List all the allowed values of $\{m_s, m_l\}$. List also all the allowed values of $\{j, m_j\}$, where $\hat{\mathbf{J}}$ denotes the total angular-momentum operator of the system. Check that the two lists are equal in length, and also that there are the same number of pairs with a given m_j in the second list as with a given $m_s + m_l$ in the first.

2. A system has two contributions to its angular momentum, both with $j = 1$. The state $|\alpha\rangle$ is defined as
$$|\alpha\rangle = \sqrt{\tfrac{1}{2}}|1,0\rangle \otimes |1,-1\rangle - \sqrt{\tfrac{1}{2}}|1,-1\rangle \otimes |1,0\rangle.$$
Show that $|\alpha\rangle$ is an eigenstate of $\hat{\mathbf{J}}^2 = \hat{J}_+\hat{J}_- + \hat{J}_3^2 - \hbar\hat{J}_3$, and find the total-angular-momentum quantum number J.
[*Hint: use* $\hat{J}_i = \hat{J}_i^{(1)} + \hat{J}_i^{(2)}$ *(strictly* $\hat{J}_i^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{J}_i^{(2)}$*).*]

3. A system has two contributions to its angular momentum: $j_1 = 3/2$ and $j_2 = 1$. These can couple up to various values of the total angular momentum, which we denote by J. Using a table from the PDG review or other source of Clebsch-Gordan coefficients:

   i) write down the expansions of the following eigenstates of total angular momentum $|J, M\rangle$ in terms of the states $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$:
   $$\text{(a)}\quad |5/2, 3/2\rangle, \qquad\qquad \text{(b)}\quad |1/2, -1/2\rangle;$$

   ii) write down the expansions of the following states of given m_1 and m_2 in terms of the eigenstates of total angular momentum $|J, M\rangle$:
   $$\text{(c)}\quad |3/2, 3/2\rangle \otimes |1, 0\rangle, \qquad\qquad \text{(d)}\quad |3/2, 1/2\rangle \otimes |1, -1\rangle.$$

   Now consider a spin-$\frac{3}{2}$ particle in a p-wave orbital.

   iii) Assume that measurements of the total angular momentum have determined its quantum numbers to be $J = \frac{1}{2}$ and $M = -\frac{1}{2}$. If the z-component of orbital angular momentum is now measured, what values of m_l may be found, and with what probabilities? In each case, what would a subsequent measurement of the z-component of the spin give?

   iv) Now assume that the z-components of the orbital and spin angular momenta have been measured and the associated quantum numbers are $m_l = -1$ and $m_s = \frac{1}{2}$. If the total angular momentum is now measured, what values of J may be found, and with what probabilities?

4. A system consists of two particles, both with $s = 1$. Using a table or other source of Clebsch-Gordan coefficients, write down the expansion of the three eigenstates of total spin angular momentum $|S, M\rangle$ with $M = 0$ in terms of the states $|s_1, m_1\rangle \otimes |s_2, m_2\rangle$. What changes if the particles are identical and have relative orbital angular momentum $l = 0$?
Challenge: find a general rule for the total spin of two identical particles with any spin.

5. In the lectures we looked at the variational calculation for the potential $V(x) = \beta x^4$. Fill in the details, using as your trial wavefunction $\Psi(x) = \mathrm{e}^{-ax^2/2}$. [*You should get an upper bound for the ground-state energy of* $(3/8)6^{1/3} = 0.68$ *in units of* $(\hbar^4\beta/m^2)^{1/3}$.]

6. A particle moves in a potential $V(x) = \beta x$ for $x > 0$ and $V(x) = \infty$ for $x < 0$. Sketch the potential and the rough form of the ground-state solution. Use an appropriate state of the harmonic oscillator as a trial function (with x_0 as a variational parameter) to obtain an upper bound on E_0. Try any other suitable trial function you can think of and see if it gives a better bound. [*Hint: the exact answer is* $2.338(\hbar^2\beta^2/2m)^{1/3}$. *If you make the wrong choice of trial function, you may get an answer below this. Think carefully about what might have gone wrong, and why the variational principle is not violated.*]

   Challenge: the solution to the differential equation $y''(z) - zy(z) + \mu y(z) = 0$ can be written in terms of Airy functions as: $y = C\mathrm{Ai}(z-\mu) + D\mathrm{Bi}(z-\mu)$. (These are special functions whose roots can be found only numerically; see the appendices to the lecture notes for a brief introduction.) Using Mathematica (or a handbook of special functions) explore these solutions, and reproduce the exact result quoted above.

7. Consider the three-dimensional problem with potential $V(r) = -\beta\mathrm{e}^{-\mu r}/r$. For finite μ, this potential has no bound-state solutions if it is too weak. (This is typically the case in 3D.) Use a trial wave function $\Psi(r) = \mathrm{e}^{-r/a}$ to find a value of β/μ, above which at least one bound state is guaranteed. What can we say in the $\mu \to 0$ limit?

8. Use the WKB approximation to find values for the energy levels of the potential in Q6.

9. In lectures, we saw that the WKB approximation gives the exact expression for the energy levels of the harmonic oscillator. Fill in the details to obtain this result given.
   [*Hint: you will need the integral* $\int \sqrt{c^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{c^2-x^2} + c^2 \arcsin(x/c)\right)$; *use the substitution* $x = c\sin\theta$ *to show this.*]

   Challenge: show that the Schrödinger equation for a particle with $l = 0$ in a spherical harmonic-oscillator well $V(r) = \frac{1}{2}m\omega^2 r^2$ is equivalent to a 1D Schrödinger equation for $u(r) = r\psi(r)$, subject to the boundary condition that $u(0) = 0$. Hence show (with negligible extra work) that the energy levels of this system in the WKB aproximation are $E = (2n + \frac{3}{2})\,\hbar\omega$. (In fact this result is also exact; you should be able to explain it in terms of your prior knowledge of the harmonic oscillator.)

10. Use the WKB approximation to show that, for two protons colliding with center-of-mass energy E, the probability of fusion is roughly $\exp[-(r_c/R_G)^{1/2}]$, where R_G is the energy-independent Gamow radius, $R_G = \hbar/(\pi^2 m_p c\alpha)$, and $r_c(E)$ is the classical distance of closest approach. You should assume that r_c is much greater than the proton radius.

    Estimate the probabilities of proton-proton fusion (a) at the surface of the Sun ($\sim$6000 K), and (b) at the centre of the Sun ($\sim 10^7$ K).