PHYS30201 Mathematical Fundamentals of Quantum Mechanics — Examples 3

1. From the definitions $\hat{J}_\pm \equiv \hat{J}_1 \pm \mathrm{i}\hat{J}_2$ and the basic commutation relations between the angular momentum operators, $\hat{J}_i$, prove the following:

$$\text{(i) } [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_3, \qquad \text{(ii) } [\hat{J}_3, \hat{J}_+] = \hbar\hat{J}_+,$$
$$\text{(iii) } \hat{\mathbf{J}}^2 = \hat{J}_+\hat{J}_- + \hat{J}_3^2 - \hbar\hat{J}_3, \qquad \text{(iv) } [\hat{\mathbf{J}}^2, \hat{J}_\pm] = 0.$$

2. In spherical polar coordinates, the angular momentum operators are

$$\hat{L}_+ \longrightarrow \hbar \mathrm{e}^{\mathrm{i}\phi}\left(\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi}\right), \quad \hat{L}_- \longrightarrow \hbar \mathrm{e}^{-\mathrm{i}\phi}\left(-\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi}\right), \quad \hat{L}_z \longrightarrow -\mathrm{i}\hbar\frac{\partial}{\partial\phi}.$$

   (a) Check that $[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm$.
   [*Hint: remember that differential operators need to act on something!*]

   (b) For a given l, the spherical harmonic with $m = l$ is proportional to $(x + \mathrm{i}y)^l$:

$$Y_l^l(\theta,\phi) = (-1)^l N_l \sin^l\theta\, \mathrm{e}^{\mathrm{i}l\phi}, \qquad \text{with} \quad N_l = \frac{\sqrt{(2l+1)!}}{\sqrt{4\pi}\, 2^l\, l!}.$$

   Use this to obtain forms for all the $l = 2$ spherical harmonics.
   [*Hint: use $Y_l^{-m} = (-1)^m (Y_l^m)^*$ to reduce the work needed.*]

   (c) Check that $\hat{L}_+ Y_l^l = 0$.

3. A particle has angular momentum $l = 1$. In the L_z basis, where $(1,0,0)^\top$ corresponds to Y_1^1, $(0,1,0)^\top$ to Y_1^0, and $(0,0,1)^\top$ to Y_1^{-1}, the angular momentum operators are

$$\hat{L}_x \underset{L_z}{\longrightarrow} \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{L}_y \underset{L_z}{\longrightarrow} \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$$

   (a) Using the differential operators in Q2, calculate $\int Y_1^{1*}\hat{L}_x Y_1^0 \mathrm{d}\Omega$, and hence verify two of the elements of the first matrix.

   (b) In Q4 of Examples 1, we found the eigenvectors of the matrix representing $\hat{L}_x$. Use these to write down the position-space representation of the state with $l = 1$ which will give zero when L_x is measured.

   (c) Similarly in Q12 of Examples 2, we found the eigenvectors of the matrix representing $\hat{L}_y$; use these to write down the position-space representation of the state which will give zero when L_y is measured. Comment on your results for (b) and (c).

4. A particle with angular momentum $l = 1$ passes through a sequence of Stern-Gerlach devices that measure components of its angular momentum. Use the eigenvectors from the previous question to find the probabilities that:

(i) the z component of angular momentum will be found to be $+\hbar$, given that the x component was previously found to be 0;

(ii) the x component will be found to be $-\hbar$, given that the z component was previously found to be $+\hbar$;

(iii) the y component will be found to be $-\hbar$, given that the x component was previously found to be $+\hbar$.

(iv) the z component will be found to be 0, given that the y component was previously found to be 0.

5.(a) Verify the commutation relations for the matrices $S_i = \frac{\hbar}{2}\sigma_i$, where σ_i are the Pauli matrices.

(b) Show that if $\mathbf{a}$ and $\mathbf{b}$ are real 3D vectors, $(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b}\,\mathbf{I} + \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b}$, where $\mathbf{I}$ is the identity matrix.

(c) Verify that any 2×2 Hermitian matrix can be written as $a_0\mathbf{I} + \mathbf{a}\cdot\boldsymbol{\sigma}$, where $a_0, \ldots, a_3$ are four real numbers.

6. Consider a spin-$\frac{1}{2}$ system.

(a) What are the eigenvalues of $\hat{S}_x$ and $\hat{S}_y$? In the S_z basis, where $\hat{S}_i \longrightarrow \frac{\hbar}{2}\sigma_i$, find the corresponding eigenvectors (the representations of $|\hat{\mathbf{x}}\pm\rangle$ and $|\hat{\mathbf{y}}\pm\rangle$).

(b) In a Stern-Gerlach experiment similar to the one depicted in Q5 of Examples 2, A and B represent measurements of components of spin along some pair of the $\{x,\ y,\ z\}$ axes (e.g. $\hat{A} = \hat{S}_x$ and $\hat{B} = \hat{S}_y$). Verify that the same probabilities are obtained for any pair of these axes.

[*Hint: you should not have to calculate more than three overlaps.*]

(c) Find the matrix representations of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in the S_x basis (where $|\hat{\mathbf{x}}+\rangle \longrightarrow \binom{1}{0}$ and $|\hat{\mathbf{x}}-\rangle \longrightarrow \binom{0}{1}$). Find also the representations of $|\hat{\mathbf{y}}\pm\rangle$ and $|\hat{\mathbf{z}}\pm\rangle$ in this basis.

[*Hint: either use the expressions in the S_z representation to find the required components, such as $\langle\hat{\mathbf{x}}\pm|\hat{S}_y|\hat{\mathbf{x}}\pm\rangle$ and $\langle\hat{\mathbf{x}}\pm|\hat{\mathbf{y}}+\rangle$ for the representations of $\hat{S}_y$ and $|\hat{\mathbf{y}}+\rangle$ respectively, or construct the unitary matrix that transforms the basis.*]

7.(a) If Q5 of Examples 2 describes measurements on a spin-$\frac{1}{2}$ particle, the operator $\hat{A}$ in the question is $(2/\hbar)\hat{S}_z$. What is $\hat{B}$?

(b) Working in the S_z representation for a spin-$\frac{1}{2}$ system, construct the matrices that represent $\hat{S}_{y'}$ and $\hat{S}_{z'}$, which are the components of the spin in the directions:

$$\mathbf{e}'_y = \cos\alpha\,\mathbf{e}_y + \sin\alpha\,\mathbf{e}_z \qquad \text{and} \qquad \mathbf{e}'_z = \cos\alpha\,\mathbf{e}_z - \sin\alpha\,\mathbf{e}_y.$$

Find $[\hat{S}_{y'}, \hat{S}_{z'}]$ and comment on your result.

8. For a spin-$\frac{1}{2}$ system:

   (a) Verify that $\langle \hat{\mathbf{n}}+|\hat{\mathbf{S}}|\hat{\mathbf{n}}+\rangle = \frac{\hbar}{2}\hat{\mathbf{n}}$ (i.e. the expectation value of the vector operator $\hat{\mathbf{S}}$ is parallel to $\hat{\mathbf{n}}$).

   (b) Find the spin-up eigenstate of $\hat{\mathbf{S}} \cdot \hat{\mathbf{n}}$ for:
   (i) $\hat{\mathbf{n}} = (-1, 0, \sqrt{3})/2$,
   (ii) $\hat{\mathbf{n}} = (1, 1, \sqrt{2})/2$?
   In each case, if the system starts in this state, find the probability of obtaining $+\hbar/2$ when a measurement of S_x is made.

9. For a spin-$\frac{3}{2}$ system, construct the matrices representing $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ in the S_z basis.
   [*Hint: find the representation of* $\hat{S}_+$ *as an intermediate step.*]