PHYS30201 Mathematical Fundamentals of Quantum Mechanics **Examples 2**

1. A particle is in a state $|\psi\rangle$ when the observable corresponding to $\hat{\Omega}$ is measured. Show that the following two statements are equivalent:

   (I) the probability of getting a result ω_i is $|\langle\omega_i|\psi\rangle|^2$;
   (II) the expectation value (ensemble average) is $\langle\psi|\hat{\Omega}|\psi\rangle$.

   [*Do NOT assume that* $|\psi\rangle$ *is an eigenstate of* $\hat{\Omega}$ *– the equivalence is trivial in that case.*]

2. At a given time, a particle is in a state $|\phi_0\rangle$, with

$$\phi_0(x) = \left(\frac{1}{\sqrt[4]{\pi a^2}}\right) \mathrm{e}^{-x^2/2a^2},$$

   and $a = \sqrt{\hbar/m\omega}$. A measurement is made of the momentum.
   What is the probability of getting an answer within a small range $\delta p = \hbar/100a$ centred on the value $\hbar/a$?

3. A hydrogen atom is prepared in an initial state described by the wave function

$$\psi_I(\mathbf{r}) = \frac{1}{\sqrt{96\pi a_0^5}}\, r \exp\left(-\frac{r}{2a_0}\right),$$

   where a_0 is the Bohr radius. A measurement is made of the energy.

   What is the probability of obtaining the ground state energy, -13.6 eV?

   What other values of the energy might be obtained?

   What would the probability be if instead the initial state were $\sqrt{3}\cos(\theta)\psi_I(\mathbf{r})$?

   [*Hint: You will need the ground-state wave function of hydrogen to answer this; it is proportional to* $\exp(-r/a_0)$. *The first answer works out as about 0.23.*]

4. In a particular orthonormal basis, the Hamiltonian of a certain system is represented by

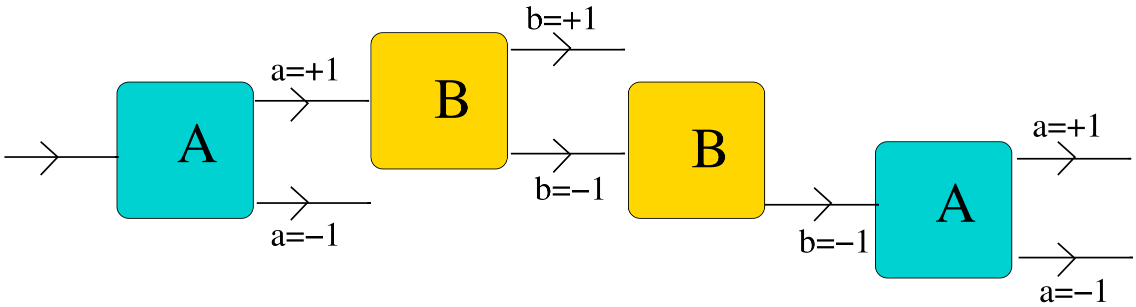
$$\hat{H} \longrightarrow \frac{\mu\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   Construct the representation of the propagator $\hat{U}(t,0)$ in the same basis, and hence find the subsequent state vectors for:

   (i) $|\psi(0)\rangle \longrightarrow (0,1,0)^\top$, (ii) $|\psi(0)\rangle \longrightarrow (1,0,-1)^\top/\sqrt{2}$.

   The eigenvectors of the matrix representation of $\hat{H}$ were obtained in Q4(i) of Examples 1. Show that your results can also be obtained by first decomposing $|\psi(0)\rangle$ in the eigenbasis of $\hat{H}$.

5.



The picture above represents a series of measurements of observables associated with operators $\hat{A}$ and $\hat{B}$, each of which has only two eigenvalues, ± 1. In the basis $\{|a+\rangle, |a-\rangle\}$ the operators are given by

$$\hat{A} \underset{a}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{B} \underset{a}{\longrightarrow} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

where θ is some real parameter. At each step, give the fraction of particles for which the measurement yields 1 or -1.

[*Hint: first show that the eigenvectors of* $\hat{B}$ *can be represented as follows:*

$$|b+\rangle \underset{a}{\longrightarrow} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad |b-\rangle \underset{a}{\longrightarrow} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix};$$

see also Q3(iv) of Examples 1.]

6. Use Ehrenfest's theorem to show that for the one-dimensional harmonic oscillator in any state (not necessarily a stationary state),

$$m\frac{\mathrm{d}^2\langle\hat{x}\rangle}{\mathrm{d}t^2} = -k_s\langle\hat{x}\rangle,$$

where k_s is the spring constant. Comment on this result.

7. Consider a system with Hamiltonian $\hat{H} = \hbar\gamma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and an observable $\hat{\Omega} = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. Check that the state $\begin{pmatrix} \cos\gamma t \\ -\mathrm{i}\sin\gamma t \end{pmatrix}$ is a solution of the time-dependent Schrödinger equation, and verify that Ehrenfest's theorem holds for the expectation value of $\hat{\Omega}$.

8. A particle of mass m is bound by a potential $V(\mathbf{r})$. Show that if the potential is spherically symmetric, $V = V(r)$, the expectation value of the angular momentum is conserved.
[*You may use the commutation relations from Q8 of Examples 1.*]

Now consider a non-spherically-symmetric potential $V = V_0(r) + zV_1(r)$. Show that $\langle\hat{L}_z\rangle$ is still conserved, and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{L}_x\rangle = -\langle\hat{y}\,\hat{V}_1\rangle \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{L}_y\rangle = \langle\hat{x}\,\hat{V}_1\rangle$$

Check that the right-hand sides correspond to the expectation values of the components of the torque, as expected.

9. This question uses the raising and lowering operators for the harmonic oscillator, $\hat{a}^\dagger$ and $\hat{a}$.

   i) Verify the numerical coefficients in the expressions $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n{+}1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n{-}1\rangle$, and show that these imply $\langle n|\hat{a} = \sqrt{n+1}\langle n+1|$ and $\langle n|\hat{a}^\dagger = \sqrt{n}\langle n-1|$.

   ii) Find the matrix elements $\langle m|\hat{x}|n\rangle$, $\langle m|\hat{p}|n\rangle$, $\langle m|\hat{x}^2|n\rangle$, $\langle m|\hat{p}^2|n\rangle$.

   iii) From the results of the previous part, find the uncertainty product $\Delta x \Delta p$ for a particle in the nth state, and comment on the result.

10. Verify that the definition

$$H_n(x) = \exp(x^2/2)\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \exp(-x^2/2)$$

does indeed generate the first few Hermite polynomials, as given in Q6 of Examples 1.

11. A state $|\lambda\rangle$ is an eigenstate of $\hat{a}$: $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$, for some complex λ. Find Δx and Δp.
[*You may use the following as a check on your results:* $\langle\hat{x}\rangle = \sqrt{2}x_0\mathrm{Re}[\lambda]$.]

Find an expression for the state $|\lambda\rangle$.
[*Hint: writing* $|\lambda\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, *find a recurrence relation between the coefficients* c_n, *namely:* $c_{n+1} = \lambda c_n/\sqrt{n+1}$. *Don't worry about normalisation initially, but leave the first constant,* c_0, *to be determined at the end.*]

12. Consider the symmetric two-dimensional harmonic oscillator with potential $\frac{1}{2}m\omega^2(x^2+y^2)$. Its energy eigenstates can be written $|n_x, n_y\rangle$ with energy $\hbar\omega(n_x + n_y + 1)$, n_x, n_y being non-negative integers.

What is the degeneracy of the state with energy $N\hbar\omega$, for positive integer N?

Show that the Hermitian operator $\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ can be written as

$$\hat{L} = \mathrm{i}\hbar(\hat{a}_y^\dagger\hat{a}_x - \hat{a}_x^\dagger\hat{a}_y),$$

and that it commutes with the Hamiltonian.

In the subspace of states with energy $3\hbar\omega$, $\{|2,0\rangle, |1,1\rangle, |0,2\rangle\}$, show that

$$\hat{L} \underset{N=3}{\longrightarrow} \sqrt{2}\hbar\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.$$

Verify that the eigenvectors of this matrix are

$$\frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2}\,\mathrm{i} \\ -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2}\,\mathrm{i} \\ -1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and find the corresponding eigenvalues. Write down the position-space representations of these eigenstates, first in terms of Cartesian coordinates (x, y) and then in polars (r, ϕ), where $x = r\cos\phi$ and $y = r\sin\phi$. Comment on your results.

13. For the symmetric three-dimensional harmonic oscillator, give expressions for the allowed energies and their degeneracies. Use them to predict the first three magic numbers in nuclei.

14. A vector space is formed from the tensor direct product of two other complex vector spaces. The first, $\mathbb{V}_q^3$, is a 3D space and the second, $\mathbb{V}_a^2$, is 2D. (Here q and a are just labels for the two spaces.) In the first space, a Hermitian operator $\hat{Q}$ has normalised eigenkets $\{|q+\rangle, |q0\rangle, |q-\rangle\}$ with eigenvalues $\{1, 0, -1\}$ respectively. Another operator $\hat{R}$ in this space has the action

$$\hat{R}|q+\rangle = \sqrt{\tfrac{1}{2}}|q0\rangle, \qquad \hat{R}|q0\rangle = \sqrt{\tfrac{1}{2}}\big(|q+\rangle + |q-\rangle\big), \qquad \hat{R}|q-\rangle = \sqrt{\tfrac{1}{2}}|q0\rangle.$$

In the second space, a Hermitian operator $\hat{A}$ has normalised eigenkets $\{|a+\rangle, |a-\rangle\}$ with eigenvalues $\{1, -1\}$ respectively. Another operator $\hat{B}$ in this space has the action

$$\hat{B}|a+\rangle = \tfrac{1}{2}|a-\rangle, \qquad \hat{B}|a-\rangle = \tfrac{1}{2}|a+\rangle.$$

In the product space, we can use the following states as an orthonormal basis:

$$\begin{array}{lll} |{+}{+}\rangle = |q+\rangle \otimes |a+\rangle, & |{+}{-}\rangle = |q+\rangle \otimes |a-\rangle, & |0{+}\rangle = |q0\rangle \otimes |a+\rangle, \\ |0{-}\rangle = |q0\rangle \otimes |a-\rangle, & |{-}{+}\rangle = |q-\rangle \otimes |a+\rangle, & |{-}{-}\rangle = |q-\rangle \otimes |a-\rangle, \end{array}$$

i) Using the bases given, write down the matrix representation of $\hat{R}$ in $\mathbb{V}_q^3$ and of $\hat{B}$ in $\mathbb{V}_a^2$. [*You should spot that you have already found the eigenvectors of these matrices in Q3(ii) and 4(i) of Examples 1.*]

ii) Which of the following states of the combined system are separable?

a) $|{+}{-}\rangle + |{-}{+}\rangle$,

b) $|{+}{+}\rangle + |{+}{-}\rangle$,

c) $|{+}{+}\rangle - \mathrm{i}|{+}{-}\rangle - \mathrm{i}|{-}{+}\rangle - |{-}{-}\rangle$,

d) $|{+}{+}\rangle - |{+}{-}\rangle + |{-}{+}\rangle + |{-}{-}\rangle$.

iii) If $|v\rangle = \sqrt{\tfrac{1}{3}}\,|{+}{-}\rangle + \sqrt{\tfrac{2}{3}}\,|0{+}\rangle$ and $|w\rangle = \sqrt{\tfrac{2}{3}}\,|{+}{-}\rangle - \sqrt{\tfrac{1}{3}}\,|0{+}\rangle$, show that $\langle v|v\rangle = 1$, $\langle w|v\rangle = 0$, and

$$\langle w|\hat{R} \otimes \hat{B}|v\rangle = \frac{1}{6\sqrt{2}}.$$

iv) In the basis above, the matrix representation of $\hat{R} \otimes \hat{B}$ has the form

$$\hat{R} \otimes \hat{B} \longrightarrow \sqrt{\tfrac{1}{8}} \begin{pmatrix} x & x & x & y & x & x \\ x & x & y & x & x & x \\ x & y & x & x & x & y \\ y & x & x & x & y & x \\ x & x & x & y & x & x \\ x & x & y & x & x & x \end{pmatrix}.$$

Find x and y. Write down an eigenket of $\hat{R}\otimes\hat{B}$ and verify that its column-vector representation is an eigenvector of the matrix above.