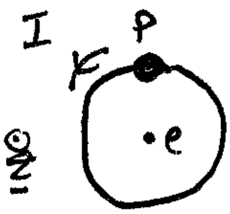
"derivations" of spin-orbit coupling

Both start from $\hat{H} = -\underline{\mu}\cdot\underline{B} = \frac{\mu_B}{\hbar} g_s \, \underline{S}\cdot\underline{B}$, $g_s = 2$.

a) Electron in its rest frame "sees" an orbiting nucleus, frequency ω, which like a current loop creates a magnetic field at the electron of $I = qf$.
ω is the electron orbital frequency, $M_e r^2 \omega = L$.

So $\underline{B} = \frac{\mu_0 I}{2r} \hat{\underline{z}} = \frac{\mu_0}{2r} \cdot \frac{Z|e|\omega}{2\pi} \hat{\underline{z}}$



$$= \frac{\mu_0 Z|e|}{4\pi r} \frac{1}{m_e r^2} \underline{L}$$

and
$$H_{so} = \frac{|e|}{M_e} \underline{S}\cdot\underline{B} = \frac{Z e^2 \mu_0}{4\pi M_e^2 r^3} \underline{L}\cdot\underline{S}$$
$$= \frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{M_e^2 c^2} \cdot \frac{1}{r^3} \underline{L}\cdot\underline{S}$$
$$= \frac{1}{M_e^2 c^2} \cdot \frac{1}{r} \cdot \frac{dV_c}{dr} \underline{L}\cdot\underline{S}$$

b) for low velocities, $\gamma \simeq 1$, $\underline{B}' = -\frac{\underline{v}\times\underline{E}}{c^2}$ ($\underline{E}$ - rest frame)

and $\underline{E} = -\hat{\underline{r}} \frac{\partial\Phi}{\partial r}$ where $-|e|\Phi = V_c$

So $\underline{v}\times\underline{E} = \frac{d\Phi}{dr} \cdot \frac{1}{r M_e} \cdot \underline{r}\times\underline{p}$

and $\underline{B} = \frac{1}{r M_e c^2 |e|} \frac{dV_c}{dr} \underline{L}$

So $H_{so} = \frac{|e|}{m_e} \underline{S}\cdot\underline{B} = \frac{1}{M_e^2 c^2} \cdot \frac{1}{r} \cdot \frac{dV_c}{dr} \cdot \underline{L}\cdot\underline{S}$

But both are out by $\frac{1}{2}$, which comes from Dirac equation