1.(a) $[\hat J_+, \hat J_-] = [\hat J_1 + \mathrm{i}\hat J_2, \hat J_1 - \mathrm{i}\hat J_2] = \mathrm{i}[\hat J_2, \hat J_1] - \mathrm{i}[\hat J_1, \hat J_2] = 2\hbar J_3$.

(b) $[\hat J_3, \hat J_+] = [\hat J_3, \hat J_1] + \mathrm{i}[\hat J_3, \hat J_2] = \mathrm{i}\hbar(\hat J_2 - \mathrm{i}\hat J_1) = \hbar \hat J_+$.

(c) First note that $\hat J_+\hat J_- + \hat J_-\hat J_+ = 2(\hat J_1^2 + \hat J_2^2)$, and also $\hat J_-\hat J_+ = \hat J_+\hat J_- - 2\hbar\hat J_3$.
Then $\hat{\mathbf{J}}^2 = \hat J_1^2 + \hat J_2^2 + \hat J_3^2 = \hat J_+\hat J_- + \hat J_3^2 - \hbar\hat J_3$.

(d) $[\hat{\mathbf{J}}^2, \hat J_+] = [\hat J_+\hat J_-, \hat J_+] + [\hat J_3^2, \hat J_+] - \hbar[\hat J_3, \hat J_+] = \hat J_+[\hat J_-, \hat J_+] + \hat J_3[\hat J_3, \hat J_+] + [\hat J_3, \hat J_+]\hat J_3 - \hbar[\hat J_3, \hat J_+]$
$= \hbar\Big(-2\hat J_+\hat J_3 + \hat J_3\hat J_+ + \hat J_+\hat J_3 - \hbar\hat J_+\Big) = \hbar\Big([\hat J_3, \hat J_+] - \hbar\hat J_+\Big) = 0.$
(Any of the other forms for $\hat{\mathbf{J}}^2$ can also be used here.)

2.(a) Acting on an arbitrary function $f(\theta,\phi)$ and using the product rule, we find that the second derivatives of f cancel between $\hat L_z\hat L_\pm f(\theta,\phi)$ and $\hat L_\pm\hat L_z f(\theta,\phi)$, leaving only the terms in which the differential operator in $\hat L_z$ acts on the functions of ϕ in $\hat L_\pm$. Since $\hat L_z \mathrm{e}^{\pm\mathrm{i}\phi} = \pm\hbar\mathrm{e}^{\pm\mathrm{i}\phi}$, we get

$$\hat L_z\hat L_\pm f(\theta,\phi) - \hat L_\pm\hat L_z f(\theta,\phi) = \pm\hbar\hat L_\pm f(\theta,\phi) \qquad \Rightarrow [\hat L_z, \hat L_\pm] = \pm\hbar\hat L_\pm.$$

(b) Using $\hat L_- Y_2^m = \hbar\sqrt{6 - m(m-1)}\,Y_2^{m-1}$, we get

$$\begin{aligned}
Y_2^2(\theta,\phi) &= \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi},\\
Y_2^1(\theta,\phi) &= \tfrac12\left(-\tfrac{\partial}{\partial\theta} + i\cot\theta\tfrac{\partial}{\partial\phi}\right)Y_2^2(\theta,\phi)\\
&= \sqrt{\frac{15}{144\pi}}\mathrm{e}^{-\mathrm{i}\phi}\left(-2\sin\theta\cos\theta\,\mathrm{e}^{2\mathrm{i}\phi} + \mathrm{i}\cot\theta(2\mathrm{i})\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}\right) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,\mathrm{e}^{\mathrm{i}\phi},\\
Y_2^0(\theta,\phi) &= -\sqrt{\frac16}\sqrt{\frac{15}{8\pi}}\left(\sin^2\theta - \cos^2\theta + \mathrm{i}\cot\theta(i)\sin\theta\cos\theta\right) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\\
Y_2^{-1}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,\mathrm{e}^{-\mathrm{i}\phi}, \qquad Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\,\mathrm{e}^{-2\mathrm{i}\phi}.
\end{aligned}$$

3. *This question is all about switching between representations. In the previous question we worked in the position-space representation, but in the subspace of states with fixed l, a $(2l+1)$-dimensional matrix representation is possible, and often useful.*

Since $\hat L_x = (\hat L_+ + \hat L_-)/2$, we can use the operators from the previous question to construct $\hat L_x$. Then, denoting the states by $|l,m\rangle$, we can calculate the matrix element $\langle 1,1|\hat L_x|1,0\rangle = \int Y_1^{1*}\hat L_x Y_1^0\,\mathrm{d}\Omega$:

$$\hat L_x Y_1^0 = \mathrm{i}\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\sqrt{\frac{3}{4\pi}}\cos\theta = -\mathrm{i}\hbar\sqrt{\frac{3}{4\pi}}\sin\phi\sin\theta,$$

$$\int Y_1^{1*}\hat L_x Y_1^0\mathrm{d}\Omega = \mathrm{i}\hbar\frac{3}{4\sqrt{2}\pi}\int_0^\pi \sin^3\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\phi}\sin\phi\,\mathrm{d}\phi = \mathrm{i}\hbar\frac{3}{4\sqrt{2}\pi}\times\frac43\times(-\mathrm{i}\pi) = \frac{\hbar}{\sqrt2},$$

which confirms the 1,2 and (via the Hermitian conjugate) the 2,1 elements of the matrix representation of $\hat L_x$.

We've already found the eigenvectors of the matrices representing $\hat{L}_x$ and $\hat{L}_y$ in the $\hat{L}_z$ basis (see Q4 of Examples 1 and Q12 of Examples 2). They have eigenvalues $\pm\hbar$ and 0 and are:

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}, \quad \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}; \qquad \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ \pm\mathrm{i}\sqrt{2} \\ -1 \end{pmatrix}, \quad \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The state which will give zero when L_x is measured is represented by $\sqrt{\frac{1}{2}}(1, 0, -1)^\top$. Expressed in position space, this is $\sqrt{\frac{1}{2}}\Big(Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)\Big) = -\sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi$. The corresponding state for $\hat{L}_y$ is $\sqrt{\frac{1}{2}}(1, 0, 1)^\top$, or $-\mathrm{i}\sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi$.

In Cartesian coordinates these are proportional to x/r and y/r respectively, which is what we should expect since the $m = 0$ eigenstate of $\hat{L}_z$ is z/r. The fact that each of these is an eigenstate of the corresponding $\hat{L}_i$ with eigenvalue zero is most easily seen in Cartesian coordinates, e.g. using $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$.

4. Denoting the eigenstates by $|\hat{\mathbf{n}}+\rangle$, $|\hat{\mathbf{n}}0\rangle$, $|\hat{\mathbf{n}}-\rangle$, where $\hat{\mathbf{n}}$ is the unit vector specifying the axis of quantisation, and using their representations from the previous question, we get

   i) $|\langle\hat{\mathbf{z}}+|\hat{\mathbf{x}}0\rangle|^2 = \big|(1,0,0)(1,0,-1)^\top/\sqrt{2}\big|^2 = 1/2$,
   ii) $|\langle\hat{\mathbf{x}}-|\hat{\mathbf{z}}+\rangle|^2 = \big|(1,-\sqrt{2},1)(1,0,0)^\top/2\big|^2 = 1/4$,
   iii) $|\langle\hat{\mathbf{y}}-|\hat{\mathbf{x}}+\rangle|^2 = \big|(1,\mathrm{i}\sqrt{2},-1)(1,-\sqrt{2},1)^\top/4\big|^2 = 1/4$,
   iv) $|\langle\hat{\mathbf{z}}0|\hat{\mathbf{y}}0\rangle|^2 = \big|(0,1,0)(1,0,1)^\top/\sqrt{2}\big|^2 = 0$.

5.(b) One way to check this is to use components to get

$$\begin{aligned}(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) &= \begin{pmatrix} (a_1 - \mathrm{i}a_2)(b_1 + \mathrm{i}b_2) + a_3b_3 & \mathrm{i}(a_2b_3 - a_3b_2) + (a_3b_1 - a_1b_3) \\ \mathrm{i}(a_2b_3 - a_3b_2) - (a_3b_1 - a_1b_3) & (a_1 + \mathrm{i}a_2)(b_1 - \mathrm{i}b_2) + a_3b_3 \end{pmatrix} \\ &= \mathbf{a}\cdot\mathbf{b}\,\mathbf{I} + \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b}.\end{aligned}$$

More elegantly, we first note that for $i \neq j$, $\sigma_i\sigma_j = -\sigma_j\sigma_i$, while for $i = j$, $\sigma_i\sigma_i = \mathbf{I}$. These can be written $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbf{I}$. This gives

$$\begin{aligned}(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) &= \tfrac{1}{2}\textstyle\sum_{ij} a_ib_j\Big(\sigma_i\sigma_j + \sigma_j\sigma_i + [\sigma_i,\sigma_j]\Big) = \tfrac{1}{2}\textstyle\sum_{ij} a_ib_j\Big(2\delta_{ij} + 2\mathrm{i}\textstyle\sum_k \epsilon_{ijk}\sigma_k\Big) \\ &= \mathbf{a}\cdot\mathbf{b}\,\mathbf{I} + \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{b},\end{aligned}$$

where we have used $(\mathbf{a}\times\mathbf{b})_k = \sum_{ij}\epsilon_{ijk}a_ib_j$. (Commonly, we drop the $\mathbf{I}$, as it is understood to multiply any scalar.)

(c) Any Hermitian matrix must have the following form, with real α, β, γ and δ:

$$\begin{pmatrix} \alpha & \beta + \mathrm{i}\gamma \\ \beta - \mathrm{i}\gamma & \delta \end{pmatrix} = \tfrac{1}{2}(\alpha+\delta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tfrac{1}{2}(\alpha-\delta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \gamma\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

and so $a_0 = \frac{1}{2}(\alpha+\delta)$, $a_1 = \beta$, $a_2 = -\gamma$ and $a_3 = \frac{1}{2}(\alpha-\delta)$.

6.(a) The eigenvalues of $\hat{S}_x$ and $\hat{S}_y$ are $\pm\frac{1}{2}\hbar$ (the same as for $\hat{S}_z$, by rotational symmetry). Knowing the eigenvalues, and using $\hat{S}_i \underset{S_z}{\longrightarrow} \frac{\hbar}{2}\sigma_i$, it is straightforward to show that

$$|\hat{\mathbf{x}}+\rangle \underset{S_z}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}1\\1\end{pmatrix},\ |\hat{\mathbf{x}}-\rangle \underset{S_z}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}1\\-1\end{pmatrix},\ |\hat{\mathbf{y}}+\rangle \underset{S_z}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}1\\\mathrm{i}\end{pmatrix},\ |\hat{\mathbf{y}}-\rangle \underset{S_z}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix}.$$

Note that these are not unique, as each can be multiplied by a different complex number of unit magnitude. These phase factors will affect the results in later questions and so, if you don't get the same results, check whether your choice of phases is the same. For the choice above, the table of overlaps is as follows (you weren't asked for this, but it will be useful below):

	$\|\hat{\mathbf{x}}+\rangle$	$\|\hat{\mathbf{x}}-\rangle$	$\|\hat{\mathbf{y}}+\rangle$	$\|\hat{\mathbf{y}}-\rangle$	$\|\hat{\mathbf{z}}+\rangle$	$\|\hat{\mathbf{z}}-\rangle$
$\langle\hat{\mathbf{x}}+\|$	1	0	$(1+\mathrm{i})/2$	$(1-\mathrm{i})/2$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\langle\hat{\mathbf{x}}-\|$	0	1	$(1-\mathrm{i})/2$	$(1+\mathrm{i})/2$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$\langle\hat{\mathbf{y}}+\|$	$(1-\mathrm{i})/2$	$(1+\mathrm{i})/2$	1	0	$1/\sqrt{2}$	$-\mathrm{i}/\sqrt{2}$
$\langle\hat{\mathbf{y}}-\|$	$(1+\mathrm{i})/2$	$(1-\mathrm{i})/2$	0	1	$1/\sqrt{2}$	$\mathrm{i}/\sqrt{2}$
$\langle\hat{\mathbf{z}}+\|$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	0
$\langle\hat{\mathbf{z}}-\|$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$\mathrm{i}/\sqrt{2}$	$-\mathrm{i}/\sqrt{2}$	0	1

(b) In the table above, all the inner products between eigenstates of different $\hat{S}_i$ have magnitude $1/\sqrt{2}$. Hence the probabilities for alternating measurements of any pair are all 50%. (In fact, since probabilities have to sum to one and in each case there are only 2 outcomes, only three matrix elements need to be calculated, say $\langle\hat{\mathbf{x}}+|\hat{\mathbf{y}}+\rangle$, $\langle\hat{\mathbf{y}}+|\hat{\mathbf{z}}+\rangle$ and $\langle\hat{\mathbf{x}}+|\hat{\mathbf{z}}+\rangle$.) This could be deduced from the results for $\hat{S}_z$ and $\hat{S}_x$ by rotational symmetry.

(c) Having constructed all the eigenvectors in the S_z representation above, it is straightforward to construct, for instance,

$$\hat{S}_z \underset{S_x}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix}\langle\hat{\mathbf{x}}+|\hat{S}_z|\hat{\mathbf{x}}+\rangle & \langle\hat{\mathbf{x}}+|\hat{S}_z|\hat{\mathbf{x}}-\rangle\\ \langle\hat{\mathbf{x}}-|\hat{S}_z|\hat{\mathbf{x}}+\rangle & \langle\hat{\mathbf{x}}-|\hat{S}_z|\hat{\mathbf{x}}-\rangle\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad |\hat{\mathbf{y}}+\rangle \underset{S_x}{\longrightarrow} \begin{pmatrix}\langle\hat{\mathbf{x}}+|\hat{\mathbf{y}}+\rangle\\ \langle\hat{\mathbf{x}}-|\hat{\mathbf{y}}+\rangle\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1+\mathrm{i}\\1-\mathrm{i}\end{pmatrix}.$$

Alternatively, we can construct the matrix $\mathbf{U}$ of eigenvectors of $\hat{S}_x$ in the S_z basis, whose elements are $\langle\hat{\mathbf{z}}\pm|\hat{\mathbf{x}}\pm\rangle$. We can then use $\mathbf{U}^\dagger\mathbf{v}$ to transform representations of kets and $\mathbf{U}^\dagger\mathbf{M}\mathbf{U}$ to transform representations of operators, for example:

$$\hat{S}_y \underset{S_x}{\longrightarrow} \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0&\mathrm{i}\\-\mathrm{i}&0\end{pmatrix}.$$

The final results, with the choices made above for the eigenvectors in the S_z basis, are

$$|\hat{\mathbf{x}}+\rangle \underset{S_x}{\longrightarrow} \begin{pmatrix}1\\0\end{pmatrix},\ |\hat{\mathbf{x}}-\rangle \underset{S_x}{\longrightarrow} \begin{pmatrix}0\\1\end{pmatrix},\ |\hat{\mathbf{y}}+\rangle \underset{S_x}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\pi/4}\\\mathrm{e}^{-\mathrm{i}\pi/4}\end{pmatrix},\ |\hat{\mathbf{y}}-\rangle \underset{S_x}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\pi/4}\\\mathrm{e}^{\mathrm{i}\pi/4}\end{pmatrix},$$

$$|\hat{\mathbf{z}}+\rangle \underset{S_x}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}1\\1\end{pmatrix},\ |\hat{\mathbf{z}}-\rangle \underset{S_x}{\longrightarrow} \sqrt{\tfrac{1}{2}}\begin{pmatrix}1\\-1\end{pmatrix},\ \hat{S}_x \underset{S_x}{\longrightarrow} \tfrac{\hbar}{2}\sigma_3,\ \hat{S}_y \underset{S_x}{\longrightarrow} -\tfrac{\hbar}{2}\sigma_2,\ \hat{S}_z \underset{S_x}{\longrightarrow} \tfrac{\hbar}{2}\sigma_1.$$

7.(a) The matrix $\mathbf{A}$ in Q5 of Examples 2 represents $(2/\hbar)\hat{S}_z$ in the usual S_z basis. In this basis, $\mathbf{B}$ represents $(2/\hbar)[\cos(2\theta)\,\hat{S}_z + \sin(2\theta)\,\hat{S}_x]$, which is proportional to the component of spin in the xz-plane at an angle 2θ to the z-axis.

(b) $\hat{S}_{y'} = \hat{\mathbf{S}}\cdot\mathbf{e}_{y'} = \cos\alpha\,\hat{S}_y + \sin\alpha\,\hat{S}_z$, $\hat{S}_{z'} = \hat{\mathbf{S}}\cdot\mathbf{e}_{z'} = -\sin\alpha\,\hat{S}_y + \cos\alpha\,\hat{S}_z$, and so

$$\hat{S}_{y'} \underset{S_z}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix} \sin\alpha & -\mathrm{i}\cos\alpha \\ \mathrm{i}\cos\alpha & -\sin\alpha \end{pmatrix}, \qquad \hat{S}_{z'} \underset{S_z}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix} \cos\alpha & \mathrm{i}\sin\alpha \\ -\mathrm{i}\sin\alpha & -\cos\alpha \end{pmatrix},$$

and

$$[\hat{S}_{y'}, \hat{S}_{z'}] = 2\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\hbar\hat{S}_x.$$

The new basis has been rotated by α about the x axis, so $\mathbf{e}'_x = \mathbf{e}_x$ and $\hat{S}_{x'} = \hat{S}_x$.

8.(a) We use $\hat{\mathbf{S}} = \hat{S}_x\,\mathbf{e}_x + \hat{S}_y\,\mathbf{e}_y + \hat{S}_z\,\mathbf{e}_z$, and we will take the terms in turn and work in the S_z basis. With $|\hat{\mathbf{n}}+\rangle \underset{z}{\longrightarrow} \begin{pmatrix} \cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix}$, we have

$$\begin{aligned}\langle\hat{\mathbf{n}}+|\sigma_x|\hat{\mathbf{n}}+\rangle &= (\cos\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi/2}, \sin\tfrac{\theta}{2}\mathrm{e}^{-\mathrm{i}\phi/2})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{-i\phi/2} \\ \sin\frac{\theta}{2}\mathrm{e}^{i\phi/2} \end{pmatrix} \\ &= \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}) = \sin\theta\cos\phi\end{aligned}$$

amd similarly $\langle\hat{\mathbf{n}}+|\sigma_y|\hat{\mathbf{n}}+\rangle = \sin\theta\sin\phi$ and $\langle\hat{\mathbf{n}}+|\sigma_z|\hat{\mathbf{n}}+\rangle = \cos\theta$.
Hence $\langle\hat{\mathbf{n}}+|\hat{\mathbf{S}}|\hat{\mathbf{n}}+\rangle = \frac{\hbar}{2}(\sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z) = \frac{\hbar}{2}\hat{\mathbf{n}}$.

(b) Using $|\hat{\mathbf{n}}+\rangle$ from above with (i) $\theta = \pi/6$, $\phi = \pi$ and (ii) $\theta = \phi = \pi/4$ gives

$$\text{(i)}, \begin{pmatrix} -\mathrm{i}\cos(\pi/12) \\ \mathrm{i}\sin(\pi/12) \end{pmatrix}, \qquad \text{(ii)} \begin{pmatrix} \cos(\pi/8)\mathrm{e}^{-\mathrm{i}\pi/8} \\ \sin(\pi/8)\mathrm{e}^{\mathrm{i}\pi/8} \end{pmatrix}.$$

In the first case, we can drop the common factor of i.
In the general case, the probability is given by $|\langle\hat{\mathbf{x}}+|\hat{\mathbf{n}}+\rangle|^2$ with $\langle x+| \longrightarrow (1,1)/\sqrt{2}$. After some straightforward work, we get

$$\tfrac{1}{2}\left|\cos\tfrac{\phi}{2}(\cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2}) - \mathrm{i}\sin\tfrac{\phi}{2}(\cos\tfrac{\theta}{2} - \sin\tfrac{\theta}{2})\right|^2 = \tfrac{1}{2}(1 + \cos\phi\sin\theta),$$

and hence for (i) 1/4 and (ii) 3/4.

9. The eigenvalues of $\hat{S}_z$ are $\hbar m$ with $m = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$. We denote the eigenvectors by $|m\rangle$, and in the S_z basis we have

$$|+\tfrac{3}{2}\rangle \underset{S_z}{\longrightarrow} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; |+\tfrac{1}{2}\rangle \underset{S_z}{\longrightarrow} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; \ldots, \qquad \hat{S}_z \underset{S_z}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Using $\hat{S}_+|m\rangle = \hbar\sqrt{\frac{15}{4} - m(m+1)}\,|m{+}1\rangle$, we can construct the elements of the matrix representing $\hat{S}_+$, for instance $\langle+\frac{3}{2}|\hat{S}_+|+\frac{1}{2}\rangle = \sqrt{3}\hbar$. Then, from $\hat{S}_- = \hat{S}_+^\dagger$, $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$ and $\hat{S}_y = -\mathrm{i}\,\frac{1}{2}(\hat{S}_+ - \hat{S}_-)$, we get

$$\hat{S}_+ \underset{S_z}{\longrightarrow} \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \hat{S}_x \underset{S_z}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \; \hat{S}_y \underset{S_z}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i}\sqrt{3} & 0 & 0 \\ \mathrm{i}\sqrt{3} & 0 & -2\mathrm{i} & 0 \\ 0 & 2\mathrm{i} & 0 & -\mathrm{i}\sqrt{3} \\ 0 & 0 & \mathrm{i}\sqrt{3} & 0 \end{pmatrix}.$$