1. (I) Starting from the probability statement, we can write the average outcome $\overline{\omega}$ (in the sense of an average over an ensemble of similarly prepared systems) as the sum of the allowed outcomes times their probabilities. These probabilities are $P_i = |\langle\omega_i|\psi\rangle|^2$, and so we get $\overline{\omega} = \sum_i P_i\omega_i = \sum_i |\langle\omega_i|\psi\rangle|^2\omega_i$.
   (II) The state $|\psi\rangle$ can be written as $|\psi\rangle = \sum_i c_i|\omega_i\rangle$, where $c_i = \langle\omega_i|\psi\rangle$. The expectation value is $\langle\psi|\hat{\Omega}|\psi\rangle = \sum_{ij} c_i^* c_j \langle\omega_i|\hat{\Omega}|\omega_j\rangle = \sum_{ij} c_i^* c_j \omega_i \delta_{ij} = \sum_i |c_i|^2\omega_i = \sum_i |\langle\omega_i|\psi\rangle|^2\omega_i$. This is the same as $\overline{\omega}$ and hence the two formulations are equivalent, and either one implies the other.

2. Let $|p\rangle$ be a momentum eigenstate. The probability of a measurement of momentum giving a result within a small range δp centred on the value p is approximately $\rho(p)\delta p$, where $\rho(p) = |\langle p|\phi_0\rangle|^2$. (For a larger range of p we would need to integrate.) We will need the well-known result that the FT of a Gaussian is another Gaussian:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{e}^{-x^2/2a^2}\mathrm{d}x &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-k^2a^2/2}\int_{-\infty}^{\infty}\mathrm{e}^{-(x+\mathrm{i}ka^2)^2/2a^2}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-k^2a^2/2}\sqrt{2\pi a^2} = a\mathrm{e}^{-k^2a^2/2}.\end{aligned}$$

   (See the relevant section of the revision/background notes for the justification of the step we have taken from the first line to the second, namely shifting x by $\mathrm{i}ka^2$ without changing the limits of integration.)

   Returning to the problem, working in the x-representation and using the result above with $k = p/\hbar$ gives

$$\langle p|\phi_0\rangle = \frac{1}{\sqrt{2\hbar\pi}}\frac{1}{\sqrt[4]{\pi a^2}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}px/\hbar}\,\mathrm{e}^{-x^2/2a^2}\mathrm{d}x = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}\,\mathrm{e}^{-p^2a^2/2\hbar^2} \text{ and so } \langle\hbar/a|\phi_0\rangle = \sqrt{\frac{a}{\hbar\mathrm{e}\sqrt{\pi}}}.$$

   Squaring this gives the probability density and so, using the narrow-interval approximation with $\delta p = \hbar/100a$, the probability is

$$P \approx \frac{a}{\hbar\mathrm{e}\sqrt{\pi}}\delta p = (100e\sqrt{\pi})^{-1} = 0.0021.$$

3. Let $|\psi_I\rangle$ be the initial state. If we measure the energy, the Postulates tell us that the probability of getting the ground-state energy, $-E_R$, is just $|\langle\psi_{100}|\psi_I\rangle|^2$, where $|\psi_{100}\rangle$ is the ground-state of hydrogen ($n = 1$, $l = 0$, $m = 0$), whose wave function is

$$\psi_{100} = Y_{00}(\theta,\phi)R_{10}(r) = \sqrt{\frac{1}{4\pi}}\frac{2}{a_0^{3/2}}\,\exp\left(-\frac{r}{a_0}\right).$$

(See the revision notes for notation.) The overlap we want is then

$$\begin{aligned}\langle\psi_{100}|\psi_I\rangle &= \frac{1}{4\sqrt{6}a_0^4\pi}\int_0^\infty\int_0^\pi\int_0^{2\pi} r\exp\left(-\frac{3r}{2a_0}\right) r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{4\pi}{4\sqrt{6}a_0^4\pi}\frac{32a_0^4}{27} = \frac{32}{27\sqrt{6}}.\end{aligned}$$

The square of this is 0.234. It is not surprising that this is quite low since, by inspection, ψ_I looks closer to the $(2,0,0)$ state; in fact the probability of getting the result $-E_{\mathrm{Ry}}/4$ is 0.75.

The state with wave function $\sqrt{3}\cos(\theta)\psi_I(\mathbf{r})$ is in a p-wave, and so has zero overlap with the ground state; $-E_{\mathrm{Ry}}$ is not a possible result of a measurement of the energy in this case. (In fact this is $|\psi_{210}\rangle$, so $-E_{\mathrm{Ry}}/4$ is the only possible result.)

4. Let's define $\hat{M} = \hat{H}/(\hbar\mu)$ (so the matrix representation $\mathbf{M}$ is $1/\sqrt{2}$ times the matrix in Q4(i) of Examples 1). We want to construct $\hat{U}(t,0) = \exp(-\mathrm{i}\hat{H}t/\hbar) \longrightarrow \exp(-\mathrm{i}\mu t\mathbf{M})$.

   One way to do this question is to note that

$$\mathbf{M} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \qquad \mathbf{M}^2 = \frac{1}{2}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}, \qquad \mathbf{M}^3 = \mathbf{M}, \quad \ldots$$

   and so the matrix $\mathbf{U}(t,0)$ can be expressed in terms of just three matrices: $\mathbf{I}$, $\mathbf{M}$ and $\mathbf{M}^2$. Doing this gives

$$\begin{aligned}\mathbf{U}(t,0) &= \exp(-i\mu t\mathbf{M}) = \mathbf{I} - \mathrm{i}\mu t\mathbf{M} - \tfrac{1}{2!}(\mu t)^2\mathbf{M}^2 - \tfrac{1}{3!}\mathrm{i}(\mu t)^3\mathbf{M}^3\ldots \\ &= \mathbf{I} + \mathrm{i}(-\mu t + \tfrac{1}{3!}(\mu t)^3 - \ldots)\mathbf{M} + (-\tfrac{1}{2!}(\mu t)^2 + \tfrac{1}{4!}(\mu t)^4 + \ldots)\mathbf{M}^2 \\ &= \mathbf{I} - \mathrm{i}\sin(\mu t)\mathbf{M} + (\cos(\mu t) - 1)\mathbf{M}^2 \\ &= \begin{pmatrix}\cos^2(\mu t/2) & -\mathrm{i}\sin(\mu t)/\sqrt{2} & -\sin^2(\mu t/2)\\ -\mathrm{i}\sin(\mu t)/\sqrt{2} & \cos(\mu t) & -\mathrm{i}\sin(\mu t)/\sqrt{2}\\ -\sin^2(\mu t/2) & -\mathrm{i}\sin(\mu t)/\sqrt{2} & \cos^2(\mu t/2)\end{pmatrix}.\end{aligned}$$

   Alternatively, we can take advantage of the fact that in Q4(i) of Examples 1 we constructed $\mathbf{S}$, the matrix of eigenvectors of $\mathbf{M}$. In this basis, we know that $\mathbf{M}_{\mathrm{diag}} = \mathbf{S}^\dagger\mathbf{M}\mathbf{S} = \mathrm{diag}\{-\mathbf{1},\mathbf{0},\mathbf{1}\}$, and hence we have $\mathbf{U}_{\mathrm{diag}} = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}\mu t}, 1, \mathrm{e}^{-\mathrm{i}\mu t}\}$. Since $\mathbf{S}$ is unitary, we can get back to the given basis using

$$\mathbf{U}(t,0) = \mathbf{S}\mathbf{U}_{\mathrm{diag}}\mathbf{S}^\dagger = \frac{1}{4}\begin{pmatrix}1&\sqrt{2}&1\\-\sqrt{2}&0&\sqrt{2}\\1&-\sqrt{2}&1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\mu t}&0&0\\0&1&0\\0&0&\mathrm{e}^{-\mathrm{i}\mu t}\end{pmatrix}\begin{pmatrix}1&-\sqrt{2}&1\\\sqrt{2}&0&-\sqrt{2}\\1&\sqrt{2}&1\end{pmatrix},$$

   which gives the same result as above.

A more general version of this method, if we know the eigenstates $|E_n\rangle$ of $\hat{H}$, is to use $\hat{U}(t,0) = \sum_n |E_n\rangle\langle E_n| \mathrm{e}^{-\mathrm{i}E_n t/\hbar}$. Using the vector representations of the eigenstates gives the matrix $\mathbf{U}(t,0)$. (Recall that $|v\rangle\langle v|$ is represented by the 3×3 matrix formed by multiplying together the column and row vectors *in that order*.)

If the power series for $\hat{H}$ doesn't simplify as it did here, diagonalisation (or equivalently the eigenvector expansion) is the only way to proceed.

For the two initial states given, (i) $|\psi(0)\rangle \longrightarrow (0,1,0)^\top$, (ii) $|\psi(0)\rangle \longrightarrow (1,0,-1)^\top/\sqrt{2}$, the subsequent states are

$$\text{(i)} \qquad |\psi(t)\rangle \longrightarrow \begin{pmatrix} -\mathrm{i}\sin(\mu t)/\sqrt{2} \\ \cos(\mu t) \\ -\mathrm{i}\sin(\mu t)/\sqrt{2} \end{pmatrix}, \qquad\qquad \text{(ii)} \qquad |\psi(t)\rangle \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The second result is easily understood by recalling that $(1,0,-1)^\top/\sqrt{2}$ is an eigenvector of $\mathbf{M}$ with eigenvalue 0, so that $\mathrm{e}^{-\mathrm{i}Et/\hbar} = 1$.

For the first result, we need to use the other two eigenstates of $\hat{H}$, $|\pm\rangle \longrightarrow (1, \pm\sqrt{2}, 1)^\top/2$, which have eigenvalues $E_\pm = \pm\hbar\mu$. In terms of these, we have:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle - |-\rangle\big) \Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\big(\mathrm{e}^{-\mathrm{i}\mu t}|+\rangle - \mathrm{e}^{\mathrm{i}\mu t}|-\rangle\big) \longrightarrow \begin{pmatrix} -\mathrm{i}\sin(\mu t)/\sqrt{2} \\ \cos(\mu t) \\ -\mathrm{i}\sin(\mu t)/\sqrt{2} \end{pmatrix}.$$

This method has clearly been quicker as a one-off (especially as we already knew the eigenvectors), but if one had repeated calculations to do, starting by constructing $\hat{U}(t,0)$ would save time in the end.

5. At the first stage, since we know nothing about the initial state, we expect to get either $a = 1$ or $a = -1$, but we do not know what the probabilities are.

   After getting $a = 1$, though, we know that the state of the system is $|a+\rangle$. The probability of obtaining $b = 1$ at the next step is $|\langle b+|a+\rangle|^2 = \cos^2\theta$, and of $b = -1$, $|\langle b-|a+\rangle|^2 = \sin^2\theta$.

   Having obtained $b = -1$, we know that the state of the system is now $|b-\rangle$, and we can get only $b = -1$ when we measure B again.

   At the final stage, the probability of obtaining $a = 1$ is $|\langle a+|b-\rangle|^2 = \sin^2\theta$, and of $a = -1$, $|\langle a-|b-\rangle|^2 = \cos^2\theta$.

   For $\theta = \pi/4$, the probabilities are $\cos^2\theta = \sin^2\theta = 1/2$, as in lectures. Not, for future reference, that if A and B correspond to measurements of spin components along different directions, the angle 2θ will give the direction of B from the z-axis in the $x - z$ plane.

6. The Hamiltonian for the harmonic oscillator, $\hat{H} = \hat{p}^2/(2m) + k_s\hat{x}^2/2$ and so, using standard commutation rules, $[\hat{x}, \hat{H}] = [\hat{x}, \hat{p}^2]/(2m) = \mathrm{i}\hbar\hat{p}/m$ and $[\hat{p}, \hat{H}] = (k_s/2)[\hat{p}, \hat{x}^2] = -\mathrm{i}\hbar k_s\hat{x}$. Since the operators have no time dependence ($\partial\hat{x}/\partial t = \partial\hat{p}/\partial t = 0$), Ehrenfest's theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle = \frac{1}{\mathrm{i}\hbar}\langle[\hat{x}, \hat{H}]\rangle = \frac{1}{m}\langle\hat{p}\rangle,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\hat{x}\rangle = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle = \frac{1}{\mathrm{i}\hbar m}\langle[\hat{p}, \hat{H}]\rangle = -\frac{k_s}{m}\langle\hat{x}\rangle.$$

This is just the equation for SHM and so the average position mirrors the classical trajectory, $\langle\hat{x}\rangle = A\cos(\omega t + \phi)$, where $\omega = \sqrt{k_s/m}$ is the classical frequency. (Note that for a stationary state, $\langle\hat{x}\rangle$ and its time-derivatives vanish.)

7. For $|\psi\rangle \longrightarrow (\cos\gamma t, -\mathrm{i}\sin\gamma t)^\top$, we first check that the Schrödinger equation is satisfied:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\cos\gamma t\\ -\mathrm{i}\sin\gamma t\end{pmatrix} = \hbar\gamma\begin{pmatrix}-\mathrm{i}\sin\gamma t\\ \cos\gamma t\end{pmatrix} = \hbar\gamma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\gamma t\\ -\mathrm{i}\sin\gamma t\end{pmatrix}.$$

Then, from the commutator $[\mathbf{\Omega}, \mathbf{H}] = (-\mathrm{i}\hbar^2\gamma)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ we get

$$\begin{aligned}\tfrac{1}{\mathrm{i}\hbar}\langle[\mathbf{\Omega}, \mathbf{H}]\rangle &= (-\hbar\gamma)\,(\cos\gamma t, \mathrm{i}\sin\gamma t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\gamma t\\ -\mathrm{i}\sin\gamma t\end{pmatrix}\\ &= \hbar\gamma(\sin^2(\gamma t) - \cos^2(\gamma t)) = -\hbar\gamma\cos(2\gamma t).\end{aligned}$$

On the other hand, taking the expectaion value of $\hat{\Omega}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{\Omega}\rangle = \frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-2\cos\gamma t\sin\gamma t) = -\frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sin(2\gamma t) = -\hbar\gamma\cos(2\gamma t).$$

and we see that Ehrenfest's theorem is satisfied.

8. By symmetry, it is enough to show that one component of $\hat{\mathbf{L}}$, say $\hat{L}_x$, commutes with $\hat{\mathbf{p}}^2$ and $V(r)$, since there is nothing special about the x direction. From the results of Q8(iv) on Examples 1, we already have $[\hat{L}_x, \hat{p}_x] = 0$, $[\hat{L}_x, \hat{p}_y] = \mathrm{i}\hbar\hat{p}_z$ and $[\hat{L}_x, \hat{p}_z] = -\mathrm{i}\hbar\hat{p}_y$, which give

$$[\hat{L}_x, \hat{\mathbf{p}}^2] = [\hat{L}_x, \hat{p}_y^2 + \hat{p}_z^2] = \mathrm{i}\hbar(\hat{p}_y\hat{p}_z + \hat{p}_z\hat{p}_y) + \mathrm{i}\hbar(-\hat{p}_y\hat{p}_z - \hat{p}_z\hat{p}_y) = 0.$$

We can also get

$$[\hat{L}_x, \hat{V}] = \hat{y}[\hat{p}_z, \hat{V}] - \hat{z}[\hat{p}_y, \hat{V}] \underset{x}{\longrightarrow} -\mathrm{i}\hbar\left(y\frac{z}{r} - z\frac{y}{r}\right)\frac{\mathrm{d}V}{\mathrm{d}r} = 0$$

(making use of Q9(v) from Examples 1). These can be expressed more elegantly using index notation:

$$\sum_j[\hat{L}_i, \hat{p}_j^2] = 2\mathrm{i}\hbar\sum_{jk}\hat{p}_j\,\epsilon_{ijk}\,\hat{p}_k = 0$$

and

$$[\hat{L}_i, \hat{V}] = \sum_{jk} \epsilon_{ijk}[\hat{x}_j \hat{p}_k, \hat{V}] \underset{x}{\longrightarrow} -\mathrm{i}\hbar \sum_{jk} \epsilon_{ijk} x_j \frac{x_k}{r}\frac{\mathrm{d}V}{\mathrm{d}r} = 0.$$

Hence, for a spherical potential, $[\hat{L}_i, \hat{H}] = 0$ and by Ehrenfest's theorem, $\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{L}_i \rangle = 0$.

For the non-spherically-symmetric (but still axially-symmetric) potential $V = V_0(r) + zV_1(r)$, only the second term in the potential fails to commute with $\hat{L}_i$. In addition, we get

$$[\hat{L}_i, \hat{z}V_1(r)] = \hat{z}[\hat{L}_i, V_1(r)] + V_1(r)[\hat{L}_i, \hat{z}] = V_1(r)[\hat{L}_i, \hat{z}]$$

Since $[\hat{L}_z, \hat{z}] = 0$, $\langle \hat{L}_z \rangle$ is still conserved (as expected) in an axially-symmetric potential. In addition, we have $[\hat{L}_x, \hat{z}] = -\mathrm{i}\hbar\hat{y}$ and $[\hat{L}_y, \hat{z}] = \mathrm{i}\hbar\hat{x}$, from which the desired results follow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{L}_x \rangle = -\langle \hat{y}\, \hat{V}_1 \rangle \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{L}_y \rangle = \langle \hat{x}\, \hat{V}_1 \rangle.$$

Classically the torque is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = -\mathbf{r} \times \nabla V$. This vanishes for a spherically symmetric potential, since $\nabla V(r)$ is proportional to the radial unit vector $\hat{\mathbf{r}}$. For the axial potential $\nabla V = \mathbf{e}_z V_1(r)$ + pieces proportional to $\hat{\mathbf{r}}$, and so $\boldsymbol{\tau} = (-y\mathbf{e}_x + x\mathbf{e}_y)V_1(r)$. Hence the result can also be written $\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\mathbf{L}} \rangle = \langle \boldsymbol{\tau} \rangle$.

9. i) The normalisations for the bras follow directly those of the kets, since the bra corresponding to $|\hat{a}n\rangle \equiv \hat{a}|n\rangle$ is given by $\langle \hat{a}n| = \langle n|\hat{a}^\dagger$, and similarly $\langle \hat{a}^\dagger n| = \langle n|\hat{a}$.

   In the lectures, we established that $\hat{a}|n\rangle = c_n|n{-}1\rangle$, where the kets are normalised, and we now need to determine c_n. Using the fact that $\hat{a}^\dagger\hat{a}$ is the number operator, we find

$$|c_n|^2 = \langle \hat{a}n|\hat{a}n\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n,$$

   and so we can take $c_n = \sqrt{n}$.

   Similarly we have $\hat{a}^\dagger|n\rangle = d_n|n{+}1\rangle$, and

$$|d_n|^2 = \langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|\hat{a}^\dagger\hat{a} + 1|n\rangle = n + 1,$$

   so we can take $d_n = \sqrt{n+1}$. If we are being careful, we need to check that our choices of phase are consistent. This can be done using $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$, which requires $c_n d_{n-1} = n$, which is indeed satisfied.

   ii) The position and momentum operators should be expressed in terms of the raising and lowering operators. Doing this gives:

$$\langle m|\hat{x}|n\rangle = (x_0/\sqrt{2})\langle m|\hat{a}^\dagger + \hat{a}|n\rangle = (x_0/\sqrt{2})(\sqrt{m}\delta_{(n+1)m} + \sqrt{n}\delta_{(n-1)m}),$$

$$\langle m|\hat{p}|n\rangle = (i\hbar/\sqrt{2}x_0)\langle m|\hat{a}^\dagger - \hat{a}|n\rangle = (i\hbar/\sqrt{2}x_0)(\sqrt{m}\delta_{(n+1)m} - \sqrt{n}\delta_{(n-1)m}),$$

$$\langle m|\hat{x}^2|n\rangle = (x_0^2/2)\langle m|(\hat{a}^\dagger)^2 + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1|n\rangle$$
$$= (x_0^2/2)\Big(\sqrt{m(m-1)}\delta_{(n+2)m} + \sqrt{n(n-1)}\delta_{(n-2)m} + (2n+1)\delta_{nm}\Big),$$

$$\langle m|\hat{p}^2|n\rangle = (-\hbar^2/2x_0^2)\langle m|(\hat{a}^\dagger)^2 + \hat{a}^2 - 2\hat{a}^\dagger\hat{a} - 1|n\rangle$$
$$= (-\hbar^2/2x_0^2)\Big(\sqrt{m(m-1)}\delta_{(n+2)m} + \sqrt{n(n-1)}\delta_{(n-2)m} - (2n+1)\delta_{nm}\Big).$$

iii) From the results of part (ii) we can get $\Delta x^2 = \langle n|\hat{x}^2|n\rangle - \langle n|\hat{x}|n\rangle^2 = x_0^2(n+\frac{1}{2})$ and $\Delta p^2 = (\hbar^2/x_0^2)(n+\frac{1}{2})$. These give $\Delta x \Delta p = (n+\frac{1}{2})\,\hbar$. For the ground state (a Gaussian) this is $\frac{1}{2}\,\hbar$, as small as possible. As n increases the wave function becomes peaked at the edges of the classically allowed region (which grows with increasing energy) and so it is not surprising that Δx increases. Furthermore as kinetic and potential energy are equal on average (both classically or in QM) we expect Δp to increase as well.

10.
$$\big(x - \tfrac{\mathrm{d}}{\mathrm{d}x}\big)\exp(-x^2/2) = 2x\exp(-x^2/2) = H_1(x)\exp(-x^2/2),$$
$$\big(x - \tfrac{\mathrm{d}}{\mathrm{d}x}\big)\,2x\exp(-x^2/2) = (4x^2-2)\exp(-x^2/2) = H_2(x)\exp(-x^2/2) \qquad \text{etc.}$$

11. First we note that, if $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$, then $\langle\lambda|\hat{a}^\dagger = \lambda^*\langle\lambda|$. Working terms of raising and lowering operators, we get $\langle\lambda|\hat{x}|\lambda\rangle = (x_0/\sqrt{2})\langle\lambda|\hat{a}^\dagger + \hat{a}|\lambda\rangle = \sqrt{2}x_0\mathrm{Re}\,[\lambda]$
and $\langle\lambda|\hat{p}|\lambda\rangle = (i\hbar/\sqrt{2}x_0)\langle\lambda|\hat{a}^\dagger - \hat{a}|\lambda\rangle = (\sqrt{2}\hbar/x_0)\mathrm{Im}\,[\lambda]$.

Similarly, making use of the expansions of $\hat{x}^2$ and $\hat{p}^2$ in Q9, we get
$\langle\lambda|\hat{x}^2|\lambda\rangle = (x_0^2/2)\big((\lambda^*)^2 + \lambda^2 + 2\lambda^*\lambda + 1\big) = x_0^2\big(2\mathrm{Re}\,[\lambda]^2 + \frac{1}{2}\big)$
$\langle\lambda|\hat{p}^2|\lambda\rangle = (-\hbar^2/2x_0^2)\big((\lambda^*)^2 + \lambda^2 - 2\lambda^*\lambda - 1\big) = (\hbar^2/x_0^2)\big(2\mathrm{Im}\,[\lambda]^2 + \frac{1}{2}\big)$
The uncertainties are thus $\Delta x = x_0/\sqrt{2}$, $\Delta p = \hbar/x_0\sqrt{2}$, and $\Delta x\Delta p = \frac{1}{2}\,\hbar$.

Let $|\lambda\rangle = \sum_n c_n|n\rangle$, with the $c_n \equiv c_n(\lambda)$ being coefficients to be determined. Then acting on this with $\hat{a}$ gives

$$\hat{a}|\lambda\rangle = \sum_{n=0}^{\infty} c_n\hat{a}|n\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}|n{-}1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle.$$

The defining property of $|\lambda\rangle$, $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$, then implies

$$\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle = \sum_{n=0}^{\infty} c_n\lambda|n\rangle.$$

Comparing coefficients we get the recurrence relation $c_{n+1} = \lambda c_n/\sqrt{n+1}$, and hence $c_n = c_0\lambda^n/\sqrt{n!}$. Using $|n\rangle = ((\hat{a}^\dagger)^n/\sqrt{n!})|0\rangle$ we have

$$|\lambda\rangle = c_0\sum_n \frac{\lambda^n}{\sqrt{n!}}|n\rangle = c_0\left(\sum_n \frac{(\lambda\hat{a}^\dagger)^n}{n!}\right)|0\rangle = c_0\,\mathrm{e}^{\lambda\hat{a}^\dagger}|0\rangle.$$

To normalise we need $\langle\lambda|\lambda\rangle = \sum_n |c_n|^2 = 1$, which implies $c_0 = \mathrm{e}^{-|\lambda|^2/2}$.

This state, which is called a *coherent state*, has an uncertainty product as low as the ground state! (Actually the ground state is a rather trivial example of such a state, with $\lambda = 0$.) It is most often used for describing light rather than mechanical oscillators. The average number of photons is $|\lambda|^2$, which is easy to show using the number operator $\hat{a}^\dagger\hat{a}$. It is a quantum state which most closely resembles a classical state, in that (among other properties) the probability of detecting n photons follows a Poisson distribution (obvious from the normalised

values of $|c_n|^2$) and subtracting a single quantum of energy doesn't change the form of the state. As a result $\delta n^2 = \langle n \rangle$, which is what we expect from Poisson statistics.

Challenge: What goes wrong if we try to find an eigenstate of $\hat{a}^\dagger$?

12. For $E = N\hbar\omega$, the quantum numbers satisfy $n_x + n_y = N - 1$, so $(n_x, n_y) = (N-1, 0), (N-2, 1) \ldots (0, N-1)$, which give N pairs in total, so the degeneracy is equal to N ($N = 1$ for the ground state, $N = 2$ for the first excited states etc).

Note that the symmetric potential implies a single scale, $y_0 = x_0 = \sqrt{\hbar/m\omega}$. In terms of raising and lowering operators, we have

$$\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \tfrac{1}{2}\mathrm{i}\hbar\Big((\hat{a}_x^\dagger + \hat{a}_x)(\hat{a}_y^\dagger - \hat{a}_y) - (\hat{a}_y^\dagger + \hat{a}_y)(\hat{a}_x^\dagger - \hat{a}_x)\Big) = \mathrm{i}\hbar(\hat{a}_y^\dagger\hat{a}_x - \hat{a}_x^\dagger\hat{a}_y)$$

where we have used the fact that the x and y operators commute. Also, writing $\hat{H}_x = \hbar\omega(\hat{a}_x^\dagger\hat{a}_x + \frac{1}{2})$ and similarly for $\hat{H}_y$, and using $[\hat{a}_x^\dagger, \hat{H}_x] = \hbar\omega\hat{a}_x^\dagger$ etc.,

$$\begin{aligned}[\hat{L}, \hat{H}] &= \mathrm{i}\hbar\Big(\hat{a}_y^\dagger[\hat{a}_x, \hat{H}_x] + [\hat{a}_y^\dagger, \hat{H}_y]\hat{a}_x - [\hat{a}_x^\dagger, \hat{H}_x]\hat{a}_y - \hat{a}_x^\dagger[\hat{a}_y, \hat{H}_y]\Big) \\ &= \mathrm{i}\hbar^2\omega\Big(-\hat{a}_y^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_x - \hat{a}_x^\dagger\hat{a}_y + \hat{a}_x^\dagger\hat{a}_y\Big) = 0.\end{aligned}$$

We need the following: $\hat{a}_x^\dagger\hat{a}_y|2,0\rangle = 0$, $\hat{a}_x^\dagger\hat{a}_y|1,1\rangle = \sqrt{2}|2,0\rangle$, $\hat{a}_x^\dagger\hat{a}_y|0,2\rangle = \sqrt{2}|1,1\rangle$. Also $\mathrm{i}\hat{a}_x\hat{a}_y^\dagger$ and $-\mathrm{i}\hat{a}_x^\dagger\hat{a}_y$ are Hermitian conjugates, so we need to construct only one of them:

$$-\mathrm{i}\hat{a}_x^\dagger\hat{a}_y \underset{N=3}{\longrightarrow} \sqrt{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{L} = \hbar(-i\hat{a}_x^\dagger\hat{a}_y + i\hat{a}_x\hat{a}_y^\dagger) \underset{N=3}{\longrightarrow} \sqrt{2}\hbar\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}.$$

The eigenvalues corresponding to the three eigenvectors $(1, \pm\sqrt{2}i, -1)^\top/2$ and $(1, 0, 1)^\top/\sqrt{2}$ are $2\hbar$, $-2\hbar$ and 0. The wave functions of the first pair of these are

$$\begin{aligned}\tfrac{1}{2}\big(\langle x| \otimes \langle y|\big)\big(|2,0\rangle \pm \sqrt{2}\mathrm{i}|1,1\rangle - |0,2\rangle\big) &= \tfrac{1}{2}\big(\phi_2(x)\phi_0(y) \pm \sqrt{2}\mathrm{i}\phi_1(x)\phi_1(y) - \phi_0(x)\phi_2(y)\big) \\ &= \mathrm{e}^{-x^2/2x_0^2}\mathrm{e}^{-y^2/2x_0^2}(2x_0^3\sqrt{\pi})^{-1}\big((2x^2 - x_0^2)/\sqrt{2} \pm 2\sqrt{2}\mathrm{i}xy - (2y^2 - x_0^2)/\sqrt{2}\big) \\ &= \mathrm{e}^{-(x^2+y^2)/2x_0^2}(x_0^3\sqrt{2\pi})^{-1}(x \pm \mathrm{i}y)^2 = (x_0^3\sqrt{2\pi})^{-1}\mathrm{e}^{\pm 2\mathrm{i}\phi}r^2\mathrm{e}^{-r^2/2x_0^2}\end{aligned}$$

and of the last,

$$\big(\phi_2(x)\phi_0(y) + \phi_0(x)\phi_2(y)\big)/\sqrt{2} = (x_0^3\sqrt{\pi})^{-1}(r^2 - x_0^2)\mathrm{e}^{-r^2/2x_0^2}$$

Using the result from last year, $\hat{L} = -i\hbar\frac{\partial}{\partial\phi}$ (that is, L_z in 3D), we see that these do indeed have $L = \pm 2\hbar$ and 0.

For the symmetric harmonic oscillator, uniquely, the Hamiltonian can be separated in either Cartesian or polar coordinates. It can be verified that the radial parts above satisfy the appropriate radial equation obtained by separation of variables in polar coordinates.

From circular symmetry, we would expect the states with $L = \pm 2$ to have the same energy, but it is not obvious why the qualitatively different state with $L = 0$ is degenerate with them. This is an example of an "accidental symmetry"; another is in the hydrogen atom where states of different l but the same n are degenerate. See Shankar Chapter 15 if you are interested.

These answers could have been written using direct-product notation (with $|n\rangle \otimes |m\rangle$ for $|n, m\rangle$ and $\hat{a}_x \otimes \hat{a}_y^\dagger$ etc). However this is a case where the labelling of the kets and operators makes it clear which space they belong in, and the direct product notation is cumbersome.

13. The extension to 3D is straightforward: the eigenstates are now $|n_x, n_y, n_z\rangle$ with energies $\hbar\omega(n_x + n_y + n_z + \frac{3}{2})$. States with energy $\hbar\omega(N + \frac{3}{2})$ have degeneracy $g = (N+2)!/(N!2!) = (N+2)(N+1)/2$ (the number of ways of dividing N indistinguishable objects into 3 sets, i.e. with 2 partitions, see the notes for PHYS20352).

N	0	1	2	3	4	5	6
g	1	3	6	10	15	21	28

The first 3 magic numbers (multiplying by 2 to account for the proton or neutron spin) are 2, 8 and 20 (note they are cumulative; the first two particles occupying the first shell, then the next 6 the second making 8 in all, etc.)

The next would be predicted to be 40, but by that point the spin-orbit interaction is as strong as the gaps between shells and so the simple model needs some modification.

14.i) The representations of the operators are

$$\hat{R} \underset{q}{\longrightarrow} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{B} \underset{a}{\longrightarrow} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvectors of $\mathbf{R}$ are those of $\mathbf{M}$ in Q4(i) of Examples 1; those of $\mathbf{B}$ are given in Q3(ii) of Examples 1.

ii) (a) $|+-\rangle + |-+\rangle = |q+\rangle \otimes |a-\rangle + |q-\rangle \otimes |a+\rangle$ is not separable.

(b) $|++\rangle + |+-\rangle = |q+\rangle \otimes \big(|a+\rangle + |a-\rangle\big)$ is separable.

(c) $|++\rangle - i|+-\rangle - i|-+\rangle - |--\rangle = \big(|q+\rangle - i|q-\rangle\big) \otimes \big(|a+\rangle - i|a-\rangle\big)$ is separable.

(d) $|++\rangle - |+-\rangle + |-+\rangle + |--\rangle \neq \big(|q+\rangle + \alpha|q-\rangle\big) \otimes \big(|a+\rangle + \beta|a-\rangle\big)$ is not separable, since we'd need $\alpha = 1$, $\beta = -1$ but $\alpha\beta = 1$.

iii) Since $|v\rangle = \sqrt{\frac{1}{3}}\,|+-\rangle + \sqrt{\frac{2}{3}}\,|0+\rangle$ and $|w\rangle = \sqrt{\frac{2}{3}}\,|+-\rangle - \sqrt{\frac{1}{3}}\,|0+\rangle$ are written in terms of an orthonormal basis (eg $\langle +-|0+\rangle = 0$), $\langle v|v\rangle$ is the sum of squares of the coefficients, $\frac{1}{3} + \frac{2}{3} = 1$. Similarly we get $\langle v|v\rangle = \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} = 0$.

$$(\hat{R}\otimes\hat{B})|v\rangle = \sqrt{\tfrac{1}{3}}\,\hat{R}|q+\rangle\otimes\hat{B}|a-\rangle + \sqrt{\tfrac{2}{3}}\,\hat{R}|q0\rangle\otimes\hat{B}|a+\rangle = \sqrt{\tfrac{1}{8}}\Big(\sqrt{\tfrac{1}{3}}\,|0+\rangle + \sqrt{\tfrac{2}{3}}\,(|+-\rangle + |--\rangle)\Big)$$

and $\langle w|\hat{R}\otimes\hat{B}|v\rangle = \sqrt{\frac{1}{8}}(\frac{2}{3} - \frac{1}{3}) = \frac{1}{6\sqrt{2}}$.

iv) Taking for example the third column, first row, we have $x = \sqrt{8}\langle 0+|\hat{R}\otimes\hat{B}|{+}{+}\rangle = 0$, and from the second column, third row, $y = \sqrt{8}\langle +-|\hat{R}\otimes\hat{B}|0+\rangle = 1$. So

$$\hat{R}\otimes\hat{B} \longrightarrow \sqrt{\tfrac{1}{8}}\left(\begin{array}{cc|cc|cc} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ \hline 0&1&0&0&0&1\\ 1&0&0&0&1&0\\ \hline 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{array}\right).$$

Ignore the horizontal and vertical lines for now.

Any vector of the form $|r_i\rangle\otimes|b_j\rangle$, where the notation indicates eigenvectors of $\hat{R}$ and $\hat{B}$, is an eigenvector of $\hat{R}\otimes\hat{B}$, for example $\frac{1}{2}(|q+\rangle - |q-\rangle)\otimes(|a+\rangle + |a-\rangle)$. This is represented by the column vector $\frac{1}{2}(1,1,0,0,-1,-1)\top$, which we can check to be an egenvector with eigenvalue 0.

Now look at the structure of this matrices indicated by the horizontal and vertical lines, so that the 6×6 matrix is written as 9 blocks, each block being a 2×2 matrix. The 9-block structure mirrors $\mathbf{R}$ while the blocks themselves are just multiples of $\mathbf{B}$. This is a general pattern when using this basis:

$$\hat{T}\otimes\hat{C} \to \left(\begin{array}{c|c|c} T_{11}\mathbf{C} & T_{12}\mathbf{C} & T_{13}\mathbf{C}\\ \hline T_{21}\mathbf{C} & T_{22}\mathbf{C} & T_{23}\mathbf{C}\\ \hline T_{31}\mathbf{C} & T_{32}\mathbf{C} & T_{33}\mathbf{C} \end{array}\right).$$

Had we reversed the order of the spaces, taking the 2-D one first, and reflected that in the ordering of the states, we would have got 4 blocks, each block being a 3×3 matrix. However if we were to take the states in any other order, the pattern would be obscured.