PHYS30201 Mathematical Fundamentals of Quantum Mechanics **Solutions 1**

1.i) $\langle n|a\rangle = \langle n|\left(\sum_m a_m|m\rangle\right) = \sum_m a_m\langle n|m\rangle = \sum_m a_m\delta_{nm} = a_n$

$\langle b|n\rangle = \left(\sum_m b_m^*\langle m|\right)|n\rangle = \sum_m b_m^*\langle m|n\rangle = \sum_m b_m^*\delta_{mn} = b_n^*$

ii) $\langle b|a\rangle = \Big(\sum_n b_n^*\langle n|\Big)\Big(\sum_m a_m|m\rangle\Big) = \sum_{nm} b_m^* a_n\langle n|m\rangle = \sum_{nm} b_n^* a_m\delta_{nm} = \sum_n b_n^* a_n$

iii) $\Big(\sum_n |n\rangle\langle n|\Big)|a\rangle = \sum_n |n\rangle\langle n|a\rangle = \sum_n |n\rangle a_n = |a\rangle$

Since $|a\rangle$ is arbitrary, the operator must be the identity: $\sum_n |n\rangle\langle n| = \hat{I}$.

iv) $\langle b|\hat{A}|a\rangle = \Big(\sum_n b_n^*\langle n|\Big)\hat{A}\Big(\sum_m a_m|m\rangle\Big) = \sum_{nm} b_n^*\langle n|\hat{A}|m\rangle a_m = \sum_{nm} b_n^* A_{nm} a_m$

v) $\langle b|\Big(\sum_{nm} A_{nm}|n\rangle\langle m|\Big)|a\rangle = \sum_{nm} A_{nm}\langle b|n\rangle\langle m|a\rangle = \sum_{nm} b_n A_{nm} a_m = \langle b|\hat{A}|a\rangle$,

using the result of the previous part. Since $\langle b|$ and $|a\rangle$ are both arbitrary, the operators between them are the same: $\sum_{nm} A_{nm}|n\rangle\langle m| = \hat{A}$.

vi) $\langle n|\hat{B}\hat{A}|m\rangle = \langle n|\hat{B}\big(\sum_k |k\rangle\langle k|\big)\hat{A}|k\rangle = \sum_k B_{nk}A_{km}$

vii) $\langle b|\hat{A}^\dagger|a\rangle = \sum_{nm} b_n^*(\hat{A}^\dagger)_{nm} a_m$ and $\langle a|\hat{A}|b\rangle^* = \left(\sum_{nm} a_m^* A_{mn} b_n\right)^* = \sum_{nm} b_n^* A_{mn}^* a_m$

Since $|a\rangle$ and $|b\rangle$ are arbitrary, comparing these gives $(\hat{A}^\dagger)_{nm} = A_{mn}^*$, the complex conjugate of the transpose. (Note that the dummy indices have been chosen to aid direct comparison of the final expressions.)

viii) Let $\hat{A}|a\rangle = |b\rangle$. Then $\langle b| = \langle a|\hat{A}^\dagger = \langle a|\hat{A}$, since $\hat{A}^\dagger = \hat{A}$, and hence $\langle a|\hat{A}\hat{A}|a\rangle = \langle b|b\rangle \geq 0$.

2.i) Using the fact that for an orthonormal basis $\langle 1|1\rangle = 1$, $\langle 1|2\rangle = 0$, etc. (and being careful to take complex conjugates in the expansion of $\langle\psi|$) we get

$$\langle\psi|\psi\rangle = |C|^2\big(\langle 1| - 2\mathrm{i}\langle 2| + (1-\mathrm{i})\langle 3|\big)\big(|1\rangle + 2\mathrm{i}|2\rangle + (1+\mathrm{i})|3\rangle\big) = 7|C|^2.$$

(In practice, we would not write out all these terms explicitly, but jump straight to $\langle\psi|\psi\rangle = |C|^2(1+4+2)$.) This implies $|C|^{-1} = \sqrt{7}$, which fixes the magnitude of C but not its phase. A simple choice is $C = 1/\sqrt{7}$, but anything differing from this by a complex phase factor is equally valid.

ii) Taking $C = 1/\sqrt{7}$, we have

$$|\psi\rangle \to \begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \\ \langle 3|\psi\rangle \end{pmatrix} = \frac{1}{\sqrt{7}}\begin{pmatrix} 1 \\ 2\mathrm{i} \\ 1+\mathrm{i} \end{pmatrix} \qquad \hat{G}|\psi\rangle \to \begin{pmatrix} \langle 1|\hat{G}|\psi\rangle \\ \langle 2|\hat{G}|\psi\rangle \\ \langle 3|\hat{G}|\psi\rangle \end{pmatrix} = \frac{1}{\sqrt{7}}\begin{pmatrix} 1-2\mathrm{i} \\ 1+2\mathrm{i} \\ 0 \end{pmatrix}.$$

iii)

$$\hat{G} \to \begin{pmatrix} \langle 1|\hat{G}|1\rangle & \langle 1|\hat{G}|2\rangle & \langle 1|\hat{G}|3\rangle \\ \langle 2|\hat{G}|1\rangle & \langle 2|\hat{G}|2\rangle & \langle 2|\hat{G}|3\rangle \\ \langle 3|\hat{G}|1\rangle & \langle 3|\hat{G}|2\rangle & \langle 3|\hat{G}|3\rangle \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that $\hat{G}$ is not Hermitian.

3.i) $\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$ with eigenvalue 1, and $\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$ with eigenvalue -1.

ii) $\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue $a+b$, and $\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue $a-b$.

iii) Here the ONLY eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue 1.

iv) $\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$ with eigenvalue 1, and $\begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}$ with eigenvalue -1.

v) $\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$ with eigenvalue $\mathrm{e}^{\mathrm{i}\theta}$, and $\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$ with eigenvalue $\mathrm{e}^{-\mathrm{i}\theta}$.

(i) and (iv) are Hermitian and so their eigenvalues are real and their eigenvectors are orthogonal.

If a and b are real, (ii) is also Hermitian and its eigenvalues are real; its eigenvectors are orthogonal in any case. (This is always true for real eigenvectors of complex symmetric marices.) (ii) is a frequently met case and the result for its eigenvectors is worth remembering!

(v) is the rotation matrix in the plane, so it is not surprising that it has no real eigenvectors. It is unitary and so, as expected, the eigenvalues have unit modulus and the eigenvectors are orthogonal.

For (iii) the characteristic equation is $(\lambda-1)^2=0$, and so there is a repeated root. However, when this is plugged into the equation for the components of the eigenvectors, only one constraint remains, namely that the lower component vanish. (Compare this with the case of the identity matrix, where there is no constraint and hence any vector is an eigenvector, although there are only two linearly-independent ones.) This matrix is neither Hermitian nor unitary, so it is not guaranteed to have two eigenvectors.

In all cases the sum of the eigenvalues is equal to the trace of the matrix, and their product to the determinant.

4.i) The characteristic equation is $\lambda(\lambda^2-2)=0$ so the eigenvalues are $-\sqrt{2}, 0, \sqrt{2}$. The corresponding eigenvectors and the matrix of eigenvectors are:

$$\tfrac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \quad \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \tfrac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}; \qquad \mathbf{S}=\tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

which is unitary. The product is

$$\mathbf{S}^\dagger\mathbf{M}\mathbf{S}=\frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}=\begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

ii) There are no off-diagonal elements in the second row or column of $\mathbf{N}$, so $(0,1,0)^\top$ is an eigenvector with eigenvalue 6. Using the result of Q3(ii) above, the other two (unnormalised) eigenvectors are $(1,0,1)^\top$ and $(1,0,-1)^\top$, with eigenvalues 6 and 4. (This illustrates an important point of technique: if you can guess an eigenvector, it is trivial to verify your guess and determine the corresponding eigenvalue at the same time. For a Hermitian matrix, if you can guess two, you can find the third by orthogonality. If you can guess one, you can simplify finding the other two by requiring them to be orthogonal to that one.)

iii) The products of $\mathbf{M}$ and $\mathbf{N}$ are

$$\mathbf{MN} = \mathbf{NM} = \begin{pmatrix} 0 & 6 & 0 \\ 6 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix} \quad \Rightarrow \; [\mathbf{M}, \mathbf{N}] = \mathbf{0}$$

This implies that $\mathbf{M}$ and $\mathbf{N}$ should have common eigenvectors, but looking at our answers we see only $(1, 0, -1)^\top$ in both lists. However the vectors $(0, 1, 0)^\top$ and $(1, 0, 1)^\top$ are degenerate eigenvectors of $\mathbf{N}$ and hence any combination of them is also an eigenvector of $\mathbf{N}$. An equally good choice for an orthogonal pair is $(1, \sqrt{2}, 1)^\top$ and $(1, -\sqrt{2}, 1)^\top$. This shows that the eigenvectors of $\mathbf{M}$ are also eigenvectors of $\mathbf{N}$, and the pair that are degenerate with respect to $\mathbf{N}$ are distinguished by their eigenvalues of $\mathbf{M}$.

5. First, we note that $\mathbf{\Omega}^2 = \mathbf{I}$, and so $\mathbf{\Omega}^3 = \mathbf{\Omega}$ etc. Then we can write

$$\begin{aligned} e^{ia\mathbf{\Omega}} &= \mathbf{I} + ia\mathbf{\Omega} - \tfrac{1}{2!}a^2\mathbf{\Omega}^2 - i\tfrac{1}{3!}a^3\mathbf{\Omega}^3 \ldots \\ &= \left(1 - \tfrac{1}{2!}a^2 + \tfrac{1}{4!}a^4 \ldots\right)\mathbf{I} + i\left(a - \tfrac{1}{3!}a^3 + \tfrac{1}{5!}a^5 \ldots\right)\mathbf{\Omega} = \cos a\,\mathbf{I} + i\sin a\,\mathbf{\Omega} \end{aligned}$$

An alternative method (which harder in this case, but more general) is to transform to a basis in which $\mathbf{\Omega}$ is diagonal, by using the matrix of its eigenvectors, $\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The eigenvalues of $\mathbf{\Omega}$ are ± 1 and so $\mathbf{S}^\dagger e^{ia\mathbf{\Omega}}\mathbf{S}$ is diagonal with elements $e^{\pm ia}$; transforming back to the original basis gives the same result as above.

6. The following results are useful for Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\pi/\alpha}, \quad \text{and} \quad \int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^2}\,dx = (-1)^n \frac{d^n}{d\alpha^n}\sqrt{\frac{\pi}{\alpha}}.$$

i) $1 = \langle 1|1\rangle = N_1^2 \int_{-\infty}^{\infty} 4x^2 e^{-x^2} = N_1^2 2\sqrt{\pi} \Rightarrow N_1 = 1/\sqrt[4]{4\pi}$.

ii) $\langle 0|2\rangle = N_0N_2 \int_{-\infty}^{\infty} \phi_0^*(x)\phi_2(x)dx = 2N_0N_2 \int_{-\infty}^{\infty}(2x^2 - 1)e^{-x^2}dx = 0$.

iii) By inspection we see that $f(x) = \phi_2(x)/(4N_2) + \phi_0(x)/(2N_0)$, and hence $f_0 = \pi^{1/4}/2$, $f_2 = \pi^{1/4}/\sqrt{2}$, $f_1 = f_3 = 0$. (Indeed $f_n = 0$ for all other n.)

iv) We can do this question with Gaussian integrals: $\langle 1|\hat{p}|0\rangle = -i\hbar \int_{-\infty}^{\infty} \phi_1(x)\frac{d}{dx}\phi_0(x)dx$ etc. An alternative (possibly easier) way is to use the orthogonality of the basis functions as follows. By explicit differentiation of the functions, we get $\hat{p}|0\rangle = i\hbar(N_0/2N_1)|1\rangle$ and $\hat{p}|1\rangle = -i\hbar\Big(N_1/N_0|0\rangle - N_1/(2N_2)|2\rangle\Big)$, and hence $\langle 1|\hat{p}|0\rangle = i\hbar N_1/N_0 = i\hbar/\sqrt{2}$ and $\langle 2|\hat{p}^2|0\rangle = -\hbar^2 N_0/(4N_2) = -\hbar^2/\sqrt{2}$.

[*By the time you get these solutions we should have met creation and annihilation operators in the lectures. Check your results for this part, noting that we have set* $x_0 = 1$.]

v) The equation for $h(x)$ is Hermite's equation with $2n$ replaced by $\mathcal{E}-1$. The finite solutions of Hermite's equation (those where the recursion relation for the coefficients of a series solution terminates) are those for which n is a non-negative integer. This means we need $\mathcal{E}-1=2n$, i.e. $\mathcal{E}$ to be a positive odd integer. (See Appendix A.4 of the lecture notes.)

7. i) $\langle x|\psi\rangle = \psi(x)$

ii) $\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}\phi^*(x')\psi(x')\,\mathrm{d}x'$

iii) $\langle\phi|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty}\phi^*(x)x\,\psi(x)\,\mathrm{d}x$

iv) $\langle x|\hat{p}|\psi\rangle = -\mathrm{i}\hbar\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}$

v) $\langle p|\psi\rangle = \int_{-\infty}^{\infty}\langle p|x'\rangle\langle x'|\psi\rangle\,\mathrm{d}x' = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}px'/\hbar}\,\psi(x')\,\mathrm{d}x'\ \big(=\Psi(p)\big)$

vi) $\langle\psi|\hat{H}|\psi\rangle = \displaystyle\int_{-\infty}^{\infty}\psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+V(x)\psi(x)\right)\mathrm{d}x$

Only (i) and (iv) are functions of position, and (v) is a function of momentum (the Fourier transform of ψ). Expressions like (i) could be written as integrals too, for example: $\langle x|\psi\rangle = \int_{-\infty}^{\infty}\langle x|x'\rangle\langle x'|\psi\rangle\mathrm{d}x' = \int_{-\infty}^{\infty}\delta(x-x')\psi(x')\mathrm{d}x' = \psi(x)$, where "$x$" is a parameter, not the integration variable. Elsewhere x and x' are used interchangeably as (dummy) integration variables.

vii) $\langle\mathbf{r}|\psi\rangle = \psi(\mathbf{r})$

viii) $\langle\phi|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi^*(\mathbf{r}')\psi(\mathbf{r}')\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$ (or $\mathrm{d}^3\mathbf{r}'$ or $\mathrm{d}V$, with a single integration sign and the infinite limits usually implied).

ix) $\langle\phi|\hat{\mathbf{x}}|\psi\rangle = \int\phi^*(\mathbf{r}')\mathbf{r}'\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'$
$= \left(\int\phi^*(\mathbf{r}')\,x'\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'\right)\mathbf{e}_x + \left(\int\phi^*(\mathbf{r}')\,y'\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'\right)\mathbf{e}_y + \left(\int\phi^*(\mathbf{r}')\,z'\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'\right)\mathbf{e}_z$

x) $\langle\mathbf{r}|\hat{\mathbf{p}}|\psi\rangle = -\mathrm{i}\hbar\nabla\psi(\mathbf{r})$

xi) $\langle\mathbf{p}|\psi\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\int\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{r}'/\hbar}\,\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'$

xii) $\langle\psi|\hat{H}|\psi\rangle = \displaystyle\int\psi^*(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+V(\mathbf{r})\psi(\mathbf{r})\right)\mathrm{d}^3\mathbf{r}.$

Here (ix) and (x) are vectors, with (x) being a vector function of position. The other functions (vii) and (xi) are both scalars. All the functions depend on three variables (the components of either position or momentum).

8.i) $\hat{B}[\hat{A},\hat{C}]+[\hat{A},\hat{B}]\hat{C} = \hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}+\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C} = -\hat{B}\hat{C}\hat{A}+\hat{A}\hat{B}\hat{C} = [\hat{A},\hat{B}\hat{C}]$

ii) First, we note that $\hat{A}\hat{B}=\hat{B}\hat{A}+c\hat{I}$, and so $\hat{B}^m\hat{A}\hat{B}^{n-m}=\hat{B}^{m+1}\hat{A}\hat{B}^{n-m-1}+c\hat{B}^{n-1}$. Using this repeatedly, we can take $\hat{A}$ through the chain of $\hat{B}$s in n steps, picking up a term $c\hat{B}^{n-1}$ at each step. This gives $\hat{A}\hat{B}^n=\hat{B}^n\hat{A}+nc\hat{B}^{n-1}$.

Alternatively, we can use induction. We first assume the result is true for some k: $[\hat{A},\hat{B}^k]=ck\hat{B}^{k-1}$. Then we can show

$$[\hat{A},\hat{B}^{k+1}]=[\hat{A},\hat{B}^k\hat{B}]=[\hat{A},\hat{B}^k]\hat{B}+\hat{B}^k[\hat{A},\hat{B}]=ck\hat{B}^{k-1}\hat{B}+c\hat{B}^k=c(k+1)\hat{B}^k;$$

hence if it holds for $n = k$ it also holds for $n = k+1$. However, we know that it is true by definition for $n = 1$, and so it must be true for all $n \geq 1$.
Using the power series for $\mathrm{e}^{\hat{B}}$, we get $[\hat{A}, \mathrm{e}^{\hat{B}}] = \sum_{n=0} \frac{1}{n!}[\hat{A}, \hat{B}^n] = \sum_{n=1} \frac{c}{(n-1)!}\hat{B}^{n-1} = c\,\mathrm{e}^{\hat{B}}$.

iii) Let $Q(x) = \sum_m q_m x^m$, so $R(x) = \sum_m m q_m x^{m-1}$. Then from the result above, with $c = -\mathrm{i}\hbar$, $[\hat{p}, \hat{Q}] = -\mathrm{i}\hbar \sum_m q_m [\hat{p}, \hat{x}^m] = -\mathrm{i}\hbar \sum_m m q_m \hat{x}^{m-1} = -i\hbar \hat{R}$.
An alternative approach is to act on be an arbitrary vector $|f\rangle$. Then, in the x-representation where $\langle x|\hat{Q}|f\rangle = Q(x)f(x)$, we get

$$[\hat{p}, \hat{Q}]|f\rangle \underset{x}{\longrightarrow} -\mathrm{i}\hbar \left(\frac{\mathrm{d}(Qf)}{\mathrm{d}x} - Q(x)\frac{\mathrm{d}f}{\mathrm{d}x} \right) = -\mathrm{i}\hbar \frac{\mathrm{d}Q}{\mathrm{d}x} f(x) = -\mathrm{i}\hbar R(x) f(x).$$

Since $|f\rangle$ is arbitrary, this implies the operator relation $[\hat{p}, \hat{Q}] = -i\hbar \hat{R}$.

iv) We can use $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$ and $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$ to reduce the compound commutators to simple ones, without ever writing expressions like $\hat{A}\hat{B} - \hat{B}\hat{A}$. Furthermore the only non-vanishing commutators among the $\hat{x}_i$ and $\hat{p}_j$ are $[\hat{p}_i, \hat{x}_i] = -\mathrm{i}\hbar \hat{I}$.
(a) $[\hat{L}_x, \hat{x}] = [\hat{L}_x, \hat{p}_x] = 0$ because $\hat{x}$ and $\hat{p}_x$ commute with all of $\hat{y}$, $\hat{p}_z$, $\hat{z}$ and $\hat{p}_y$.
(b) $[\hat{L}_x, \hat{y}] = [\hat{y}\,\hat{p}_z, \hat{y}] - [\hat{z}\,\hat{p}_y, \hat{y}] = 0 - \hat{z}[\hat{p}_y, \hat{y}] - [\hat{z}, \hat{y}]\hat{p}_y = i\hbar \hat{z}$.
(c) $[\hat{L}_x, \hat{p}_z] = -[\hat{z}\,\hat{p}_y, \hat{p}_z] = -[\hat{z}, \hat{p}_z]\hat{p}_y = -i\hbar \hat{p}_y$.
The full set of relations like these can be written

$$[\hat{L}_i, \hat{x}_j] = i\hbar \sum_k \epsilon_{ijk}\hat{x}_k, \qquad [\hat{L}_i, \hat{p}_j] = i\hbar \sum_k \epsilon_{ijk}\hat{p}_k,$$

where ϵ_{ijk} is the 3D antisymmetrix symbol: $\epsilon_{ijk} = 1$ if $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$, -1 if it is an anticylic permutation (e.g. $\{2, 1, 3\}$), and 0 if any two indices are the same.

v) In the position representation we have (as in part (iii))

$$[\hat{\mathbf{p}}, V(\hat{\mathbf{x}})]|f\rangle \underset{\mathbf{x}}{\longrightarrow} -\mathrm{i}\hbar\Big(\nabla\big(V(\mathbf{r})f(\mathbf{r})\big) - V(\mathbf{r})\nabla f(\mathbf{r})\Big) = \big(-\mathrm{i}\hbar\nabla V(\mathbf{r})\big)f(\mathbf{r}).$$

Again, since $|f\rangle$ is arbitrary, the relation must hold for the operators and so we get $[\hat{\mathbf{p}}, V(\hat{\mathbf{x}})] \underset{x}{\longrightarrow} -i\hbar\,\nabla V(\mathbf{r})$.
For $V = V(r)$, the chain rule gives

$$\nabla V = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} V(r) = \sum_i \mathbf{e}_i \frac{\partial r}{\partial x_i}\frac{\mathrm{d}V(r)}{\mathrm{d}r} = \sum_i \mathbf{e}_i \frac{x_i}{r}\frac{\mathrm{d}V(r)}{\mathrm{d}r} = \hat{\mathbf{r}}\frac{\mathrm{d}V}{\mathrm{d}r}.$$

9. For $\mathbf{p}_0 = \big(2\,\mathbf{e}_x - \mathbf{e}_z\big)\hbar/a$, we have

$$\langle \mathbf{r}|\mathbf{p}_0\rangle = \left(\tfrac{1}{2\pi\hbar}\right)^{3/2}\mathrm{e}^{\mathrm{i}\mathbf{p}_0\cdot\mathbf{r}/\hbar} = \left(\tfrac{1}{2\pi}\right)^{3/2}\mathrm{e}^{\mathrm{i}(2x-z)/a},$$

$$\langle \mathbf{p}|\mathbf{p}_0\rangle = \delta(\mathbf{p} - \mathbf{p}_0) = \delta(p_x - 2\hbar/a)\delta(p_y - 0)\delta(p_z + \hbar/a).$$

Note that both of these are scalar functions.