PHYS30201 Mathematical Fundamentals of Quantum Mechanics 2016-17: Examples 1

*The questions marked * are problems which everyone should attempt. The rest are proofs that fill in some of the more important gaps in the notes (or in one case (13) a harder example).*

1. * (i) Show that $\mathbb{C}^3$, the set of all complex number triplets, with addition and multiplication defined as usual for a row vector, is a complex vector space.
   (ii) Consider the set of all "arrow" vectors in the plane, without reference to any coordinate system. Show using representative drawings that the set satisfies all the rules of a real vector space. What is the zero vector? Draw an arbitrary vector and its inverse vector.

2. * Do the following form a real vector space, according to the mathematical definition?
   (i) The set of real functions $f(x)$ defined where $0 \le x \le L$, with $f(0) = f(L) = 0$.
   (ii) The set of real functions which are periodic in L, i.e. $f(0) = f(L)$.
   (iii) The set of real functions for which $f(0) = 4$.

3. * Consider the set of all third-order polynomials in a real variable x, with complex coefficients, defined for $-1 \le x \le 1$. Show that the set satisfies all the rules of a complex vector space. What is the zero vector? The inverse vector corresponding to $(3 + ix + (1 - 2i)x^3)$? What is the dimension of this space? Write down two possible set of basis vectors. What changes if only odd functions are included?

4. In an arbitrary vector space,
   i) show that the zero vector $|0\rangle$ is unique. (Assume that there is a second vector $|b\rangle$ which also satisfies $|v\rangle + |b\rangle = |v\rangle$, and show that $|b\rangle = |0\rangle$.)
   ii) show that $0|v\rangle = |0\rangle$ for any $|v\rangle$.

5. * Classify each of these two sets of vectors in $\mathbb{R}^3$ as linearly dependent or independent:

   $$\text{a) } \{(1,1,0),\ (1,0,1),\ (3,2,1)\}. \qquad \text{b) } \{(1,1,0),\ (1,0,1),\ (0,1,1)\}.$$

   Find the coordinate of the vector $(2, 4, 6)$ in the linearly independent set (note that at this point you should not use scalar products).

6. Show that for an N-dimensional vector space, $\mathbb{V}^N$, any set of N *linearly independent* vectors forms a basis. (Hint: look carefully at the definitions of "N-dimensional" and "basis", and show that assuming that the set is not a basis leads to a contradiction.)

7. (i)* Which of the following are valid subspaces of real 3D space, where vectors $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ are considered as dispacements from the origin:
   (a) The positive x axis; (b) the plane $z = 1$; (c) the plane $x + y + z = 0$?
   (ii) Show that the set of all vectors orthogonal to a given vector $|v\rangle$ in $\mathbb{V}^N$ form an $N - 1$ dimensional subspace.

8. (i) Consider a product space $\mathbb{V}^N \otimes \mathbb{V}^M$. Use the result of qu. 4 to show that $|v\rangle \otimes |0\rangle$ and $|0\rangle \otimes |w\rangle$ are equal to one another and to the null vector in the product space. Write down one or more equivalent expressions for the inverse of $|v\rangle \otimes |w\rangle$.
   (ii)* Consider the following two vector spaces: odd 3rd order polynomials in x and odd 3rd order polynomials in y. What is the dimension of the individual and product space? Write down examples of seperable and inseperable members of the product space.

9. * The kets $|a\rangle$ and $|b\rangle$ are represented in a certain orthonormal basis by $\begin{pmatrix} -2 \\ 2i \end{pmatrix}$ and $\begin{pmatrix} 2+3i \\ 2i \end{pmatrix}$ respectively. Find the numerical values of $\langle a|b\rangle$ and $\langle b|a\rangle$, and the norm of $|c\rangle = |a\rangle + |b\rangle$.

10. * $\{|1\rangle, |2\rangle, |3\rangle\}$ are orthonormal vectors. Let $|\psi\rangle = C\big(|1\rangle + 2i|2\rangle + (1+i)|3\rangle\big)$. Find the constant C if $|\psi\rangle$ is to be normalised.

11. * From the set in qu. 5 which is linearly independent, form an orthonomal basis by Gram-Schmidt orthogonalisation

12. Prove the Schwarz and triangle inequalities:

    i) $|\langle b|a\rangle| \le |a|\ |b|$. (Hint: defining $|\phi\rangle = |a\rangle - C|b\rangle$, where $C = \langle b|a\rangle/\langle b|b\rangle$, use the fact that $\langle\phi|\phi\rangle$ must be non-negative.)

    ii) $|a+b| \le |a| + |b|$. (Hint: as both sides are non-negative, consider the squares of both sides; then use the Schwarz inequality.)

13. Consider the set of all 2×2 complex matrices, with the sum defined in the usual way, and with the inner product of two matrices $\mathbf{M}$ and $\mathbf{N}$ being defined as $\frac{1}{2}\mathrm{Tr}(\mathbf{M}^\dagger\mathbf{N})$. *Note that in this question matrices are (abstract) vectors, not operators on vectors.* (Note $\mathrm{Tr}(\alpha\mathbf{A}+\beta\mathbf{B}) = \alpha\mathrm{Tr}(\mathbf{A}) + \beta\mathrm{Tr}(\mathbf{B})$ (linearity), $\mathrm{Tr}(\mathbf{AB}\ldots\mathbf{MN}) = \mathrm{Tr}(\mathbf{NAB}\ldots\mathbf{M})$ and $\mathrm{Tr}(\mathbf{A}^\dagger) = \mathrm{Tr}(\mathbf{A})^*$.)
    i) Show that the set satisfies all the rules of a complex vector space; write down the zero vector, and the inverse vector of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (Note the latter is NOT the multiplicative matrix inverse, but the inverse under addition).
    ii) Show that the inner product satisfies all the required rules.
    iii) Show that the following matrices form as orthonormal basis in the space:

    $$|1\rangle = \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad |2\rangle = \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad |3\rangle = \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad |4\rangle = \sqrt{\tfrac{1}{2}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

    Find the coefficients m_i of $|M\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in this basis, and verify that $\langle M|M\rangle = \sum_i |m_i|^2$.

14. * Let $\{|i\rangle\}$ be an orthonormal basis of $\mathbb{V}^N$. Let a_i be the ith coordinate of $|a\rangle$ in this basis, and b_i that of $|b\rangle$, so that $|a\rangle = \sum_i a_i|i\rangle$ and $|b\rangle = \sum_i b_i|i\rangle$. $\hat{A}$, $\hat{B}$ etc are operators in the space. Prove the following:

    i) $a_i = \langle i|a\rangle$ and $b_i^* = \langle b|i\rangle$

    ii) $\langle b|a\rangle = \sum_i b_i^* a_i = \langle a|b\rangle^*$

    iii) $\sum_i |i\rangle\langle i| = \hat{I}$. (Hint: show that acting on an arbitrary ket $|a\rangle$ gives $|a\rangle$ again.)

    iv) $\langle b|\hat{A}|a\rangle = \sum_{ij} b_i^* A_{ij} a_j$, where $A_{ij} = \langle i|A|j\rangle$.

    v) $\sum_{ij} A_{ij}|i\rangle\langle j| = \hat{A}$. (Hint: consider its action on an arbitrary ket $|a\rangle$.)

    vi) $\langle i|\hat{B}\hat{A}|k\rangle = \sum_j B_{ij}A_{jk}$. (Hint: insert the identity operator between $\hat{B}$ and $\hat{A}$. Watch the indices!)

    vii) Defining $\hat{C}$ so that $\langle b|\hat{A}|a\rangle = \langle a|\hat{C}|b\rangle^*$, show that $C_{ij} = A_{ji}^*$. (By definition $\hat{C} = \hat{A}^\dagger$, so this demonstrates that the adjoint operator is represented by the Hermitian conjugate matrix.)

15. * A certain operator $\hat{G}$ acts on vectors in a 3-D vector space and has the following effects on a certain orthonormal basis set $\{|1\rangle, |2\rangle, |3\rangle\}$:

$$\begin{aligned} \hat{G}|1\rangle &= |1\rangle + |2\rangle \\ \hat{G}|2\rangle &= -|1\rangle + |2\rangle \\ \hat{G}|3\rangle &= 0 \end{aligned}$$

Write down the matrix representation of $\hat{G}$ in this basis. Which column vectors represent $|\psi\rangle$ and $\hat{G}|\psi\rangle$, where $|\psi\rangle$ is the vector from qu. 10?

16. * For each of the following matrices, state whether it is Hermitian, unitary, both, or neither:

$$\text{(i)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \text{(ii)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \text{(iii)} \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}; \qquad \text{(iv)} \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$

17. Let $\hat{U}$ be a unitary operator and $\hat{A}$ a Hermitian one on $\mathbb{V}^N$:

 i) * Show that for any ket $|v\rangle$ in $\mathbb{V}^N$, $\langle v|\hat{A}^2|v\rangle \geq 0$.

 ii) Show that $\hat{U}\hat{A}\hat{U}^\dagger$ and $\hat{U}^\dagger\hat{A}\hat{U}$ are Hermitian;

 iii) Show that if the kets $\{|u_i\rangle\}$ form an orthonormal basis, so do $\{|v_i\rangle\}$, where $|v_i\rangle = \hat{U}^\dagger|u_i\rangle$.

 iv) Show that $\det(\hat{U}\hat{A}\hat{U}^\dagger) = \det(\hat{A})$ and $\mathrm{Tr}(\hat{U}\hat{A}\hat{U}^\dagger) = \mathrm{Tr}(\hat{A})$. (You may need to remind yourself of the properties of traces and determinants of products, some are given in qu. 13 above.)

18. A Hermitian operator on $\mathbb{V}^N$, $\hat{\Omega}$, has non-degenerate eigenvalues $\{\omega_i\}$ and normalised eigenvectors $\{|\omega_i\rangle\}$. Prove that:

 i) the eigenvalues are real and the eigenvectors are orthogonal.

 ii) $\sum_i \omega_i|\omega_i\rangle\langle\omega_i| = \hat{\Omega}$. (Hint: show that both sides give the same result when acting on an arbitrary ket $|a\rangle$, which you should first write in terms of the basis $\{|\omega_i\rangle\}$.)

19. $\hat{U}$ is a unitary operator on $\mathbb{V}^N$.

 i) From $\hat{U}^\dagger\hat{U} = \hat{I}$, prove that in any orthonormal basis the columns of the corresponding matrix $\mathbf{U}$ are orthonormal vectors, and from $\hat{U}\hat{U}^\dagger = \hat{I}$ prove the same for the rows.

 ii) If $\hat{U}$ has non-degenerate eigenvalues $\{\omega_i\}$ and normalised eigenvectors $\{|\omega_i\rangle\}$, prove that the eigenvalues have unit norm and the eigenvectors are orthogonal.

 iii) How many *real* parameters are needed to uniquely define a complex $N \times N$ unitary matrix? (Hint: One complex number needs two real numbers to define it. You might want to start by satisfying yourself that the number is N^2 for a Hermitian matrix.)

20. * Find the eigenvalues and eigenvectors of

$$\text{i)} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{ii)} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{iii)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{iv)} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \quad \text{v)} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where θ is real. Comment on your results.

21. * Find the eigenvalues and eigenvectors of $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Verify that the matrix of normalised eigenvectors $\mathbf{S}$ is unitary (equivalently that the eigenvectors are orthogonal) and that $\mathbf{S}^\dagger\mathbf{M}\mathbf{S}$ is diagonal.

22. * Find the eigenvalues and eigenvectors of $\mathbf{N} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 5 \end{pmatrix}$ (Hint: $(0,1,0)^\top$ is obviously one of them, and the other two must be orthogonal to it, of the form $(a,0,b)^\top$, so in fact you only need to consider $\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ to find $(a,b)^\top$.)
Vertify that $\mathbf{N}$ commutes with $\mathbf{M}$ from the previous question. If, as is likely, you have not obtained the same set of eigenvectors for both, explain why the theorem that a common set of eigenvectors can be found is not violated.

23. * Let $\mathbf{S} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Verify that $\mathbf{S}$ is unitary. Working in ordinary real space, in the plane, show that this is the transformation matrix between the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$, being unit vectors in the x and y directions, and basis vectors which are rotated by θ anticlockwise with respect to these, $\mathbf{e}'_1 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2$ and $\mathbf{e}'_2 = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2$ (defining $S_{ij} = \langle e_i | e'_j \rangle$). If another vector $\mathbf{a}$ has coordinates $(\cos\alpha, \sin\alpha)$ in the first basis, show that it has coordinates $\big(\cos(\alpha-\theta), \sin(\alpha-\theta)\big)$ in the second basis. Draw a diagram to illustrate these vectors. Show also that, viewed as an active transformation, $\mathbf{S}\mathbf{a}$ is a new vector rotated by θ anticlockwise with respect to $\mathbf{a}$.

24. Use the fact that Hermitian and unitary matrices can be diagonalised by a unitary transformation to provide a simple proof that the trace and determinant of such a matrix are the sum and product of the eigenvalues. (The result is more general though.)

25. * If $\mathbf{\Omega} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, show that $e^{ia\mathbf{\Omega}} = \cos a\,\mathbf{I} + i \sin a\,\mathbf{\Omega}$. (Hint: there is no need to diagonalise here, just use the power series directly.)

26. * Let $\{|1\rangle, |2\rangle\}$ be an orthonormal basis in $\mathbb{V}^2$, and let $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the representation of an operator $\hat{\Omega}$ in this basis. We can form a product space from two copies of this space $\mathbb{V}^2 \otimes \mathbb{V}^2$ (we omit labels since the order is enough to distinguish the two). Write down four orthonormal basis vectors of this product space. In this basis, calculate the matrix elements and hence write down the matrix representations of the following operators: $\hat{\Omega} \otimes \hat{I}$, $\hat{I} \otimes \hat{\Omega}$ and $\hat{\Omega} \otimes \hat{\Omega}$, and show that the third is the matrix product of the first two. Based on your results from qu. 20(ii) (with $a = 0$), write down one eigenvector of $\hat{\Omega} \otimes \hat{\Omega}$, and express it in terms of the four basis vectors of the product space.