Questions 1 and 2 are on the liquid drop model, nuclear sizes and Coulomb energies. Questions 3 and 4 are on the use of atomic masses and mass excesses to calculate Q values. Questions 5 to 7 make use of the semi-empirical mass formula.

1. A very simple model for the charge distribution of a nucleus treats it as a uniform sphere of radius R and total charge Z (in units of e):

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r < R \\ 0 & \text{for } r > R \end{cases} .$$

   (a) Find an expression for the interior density ρ_0 in terms of Z and R.

   (b) The rms (root-mean-square) radius, r_c, is defined by

$$r_c^2 = \frac{1}{Z} \int r^2 \rho(r) \, \mathrm{d}^3\mathbf{r} .$$

   Determine r_c for the uniform sphere.

   (c) Use the empirical fit to measured charge radii,

$$r_c \simeq (0.94 \text{ fm}) \, A^{1/3},$$

   to deduce a value for the constant R_0 in the liquid-drop formula

$$R = R_0 \, A^{1/3}.$$

   (d) A more realistic model for the charge density of the nucleus has a smooth surface, for example:

$$\rho(r) = \rho_c \frac{1}{1 + \exp[(r - R)/a]} .$$

   Sketch this density (assuming that a is much smaller than R) and, without trying to do any integration, explain why it gives a larger rms charge radius than a uniform sphere with the same R.

2. Show that the electrostatic energy of a uniform charged sphere is

$$V = \frac{3}{5} \frac{Z^2 \, e^2}{4\pi\epsilon_0 \, R} ,$$

   where Ze is the total charge of the sphere and R is its radius.
   [*Hint:* imagine building up the sphere from thin spherical shells and calculate the energy of a shell of radius r and thickness $\mathrm{d}r$ due to the charge already inside r.]

   Using the liquid-drop model with R_0 from question 1(a), find the coefficient a_c of the Coulomb term in the semi-empirical mass formula ($+a_c \, Z^2/A^{1/3}$). How does your result compare with values obtained from fits to data, which typically lie in the range $a_c = 0.70$ to 0.72 MeV?

3. In β^+ decay, a nucleus emits a positron and a neutrino:

$$ {}^A_Z X \rightarrow {}_{Z-1}^{\;\;A} Y + e^+ + \nu. $$

Ignoring the mass of the neutrino and the electronic binding energies, show that the energy release is

$$ Q = \big[M(Z,A) - M(Z-1,A) - 2m_e\big]c^2, $$

where $M(Z,A)$ and $M(Z-1,A)$ are the *atomic* masses of the isobars involved.

^{15}O decays to ^{15}N, emitting positrons with a maximum kinetic energy of 1.7322 MeV. ^{15}N has a mass excess of 101.4 keV$/c^2$. Determine the atomic mass of ^{15}O in atomic mass units. You should use $M_u = 931.49$ MeV$/c^2$ and you should give your answer to 9 significant figures.

4. ^{20}F decays to an excited state of ^{20}Ne, emitting electrons with a maximum (kinetic) energy of 5.391 MeV. The excited state then decays to the ground state of ^{20}Ne, emitting a photon of energy 1.63367 MeV. Given that the mass excess of ^{20}Ne is -7041.931 keV$/c^2$, find the atomic mass of ^{20}F. Give your answer in atomic mass units, showing the maximum number of digits that you trust.

5. The semi-empirical mass formula can be written in the form

$$ M(Z,A) = Z\,M({}^1\text{H}) + (A-Z)M_n - B(Z,A), $$

where the binding energy is

$$ B(Z,A) = a_v\,A - a_s\,A^{2/3} - a_c\,\frac{Z(Z-1)}{A^{1/3}} - a_a\,\frac{(A-2Z)^2}{A} + \delta_p, $$

and the pairing energy is

$$ \delta_p = \begin{cases} +a_p\,A^{-1/2} & \text{even-even} \\ 0 & \text{odd } A \\ -a_p\,A^{-1/2} & \text{odd-odd} \end{cases}. $$

Estimate the binding energy of ^{20}Ne, using the following values for the parameters, all in MeV: $a_v = 15.85$, $a_s = 18.34$, $a_c = 0.71$, $a_a = 23.21$ and $a_p = 12.0$.

Hence estimate the mass excess of ^{20}Ne and compare your answer with the measured value of -7041.931 keV$/c^2$.
[*Hint:* do not attempt to find the mass $M(Z,A)$ and subtract $A\,M_u$ from it, as that would require values for masses with unreasonable numbers of significant figures. Instead, you should work in terms of $M({}^1\text{H}) - M_u = 7.29$ MeV and $M_n - M_u = 8.07$ MeV.]

6. Use the semi-empirical mass formula

$$M(A,Z) = Z\,M(^1\mathrm{H}) + (A-Z)M_n - a_v\,A + a_s\,A^{2/3} + a_c\,\frac{Z(Z-1)}{A^{1/3}} + a_a\,\frac{(A-2Z)^2}{A} - \delta_p.$$

to show that the isobar with the lowest mass for a given odd value of A has

$$Z = \frac{M_n - M(^1\mathrm{H}) + a_c\,A^{-1/3} + 4a_a}{2a_c\,A^{-1/3} + 8a_a\,A^{-1}}.$$

Hence show that, for light atoms, the line of stability is given by $N \simeq Z$ while, for heavier atoms, it lies in the region $N > Z$. You should take the values $a_c = 0.71$ MeV and $a_a = 23.21$ MeV for the Coulomb and asymmetry parameters. [*Hint:* consider the relative sizes of the terms in the numerator, treating the mass difference $M_n - M(^1\mathrm{H})$ as one term.]

7. The isobars ^{15}O and ^{15}N form a "mirror" pair, with the numbers of protons and neutrons swapped. Find the difference in MeV between their binding energies, using your work on question 3. Explain why this difference can be attributed to their Coulomb energies alone.

   Using the expression in question 2 for the Coulomb energy of a uniformly charged sphere, estimate the radius of nuclei with $A = 15$. [Since these nuclei have fairly small values of Z, you could try to improve your estimate by replacing Z^2 by $Z(Z-1)$ in the Coulomb energy.] Compare your result with the liquid-drop model.