DISCRETE SYMMETRIES IN NUCLEAR AND PARTICLE PHYSICS

Discrete symmetries are ones that do not depend on any continuous parameter. The classic example is reflection in a mirror – either we reflect an object in the mirror or we leave it unchanged. This is in contrast to continuous symmetries like rotations where we can turn the object through any angle.

There are three discrete symmetries that play crucial roles in particle physics: parity (P), charge conjugation (C), and time reversal (T). This note collects together some of their main features.

Parity

The most familiar of these symmetries is parity, which is also important in nuclear physics and many other areas of quantum physics. It describes the effect on a system of reversing all coordinates: $x \to -x$, $y \to -y$ and $z \to -z$, or more compactly: $\mathbf{r} \to -\mathbf{r}$. In terms of spherical polar coordinates, the transformation is: $r \to r$, $\theta \to \pi - \theta$ and $\phi \to \pi + \theta$.

This can also be thought of as reflection in a mirror, $z \to -z$, followed by a rotation through 180° in the xy plane. If the system is symmetric under rotations about the z axis, the rotation has no effect. Since this is often the case for systems of interest, you may hear the parity transformation referred to as "reflection in a mirror".

As well as reversing the coordinates of all particles, the parity transformation reverses their motions, and so momenta are transformed similarly: $\mathbf{p} \to -\mathbf{p}$.

In contrast, angular momenta are invariant under this transformation: $\mathbf{J} \to \mathbf{J}$. In the case of orbital angular momentum, this follows from the definition, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, which shows that reversing both $\mathbf{r}$ and $\mathbf{p}$ leaves $\mathbf{L}$ unchanged. Alternatively, imagine a gyroscope spinning with its axis normal to a mirror; the sense of rotation of the mirror image is the same as that of the real thing. Quantities like angular momentum whose direction lies along an axis of rotation and so are unchanged under parity are known as "axial vectors" or "pseudovectors".

In quantum mechanics, we can define a parity operator $\widehat{P}$ which, acting on a single-particle wave function, has the effect

$$\widehat{P}\psi(\mathbf{r}) = \psi(-\mathbf{r}). \tag{1}$$

An eigenstate of this operator satisfies

$$\widehat{P}\psi(\mathbf{r}) = P\psi(\mathbf{r}), \tag{2}$$

where P is its eigenvalue. Since two reflections take the system back to where it started, the eigenvalue must satisfy $P^2 = 1$. Its possible values are thus $P = \pm 1$. We say that functions with $P = +1$ have even parity while those with $P = -1$ have odd parity (since $\psi(-\mathbf{r}) = P\psi(\mathbf{r})$).

A wave function with definite orbital angular momentum also has definite parity. For example,

$$Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \tag{3}$$

is obviously even under a parity transformation whereas

$$Y_{1,+1}(\theta, \phi) = -\sqrt{\frac{3}{4\pi}} \sin\theta \, \mathrm{e}^{\mathrm{i}\phi} \tag{4}$$

is odd. More generally, any wave function with even l has $P = +1$ whereas one with odd l has $P = -1$. (This is most easily checked for the cases with $m = 0$ which are even or odd polynomials of $\cos\theta$.) This means that for a single particle with orbital angular momentum l we can write

$$P = (-1)^l. \tag{5}$$

Most quantum numbers are additive: for a system of several particles we simply add the quantum numbers for the individual particles to get the total for the systems (remembering that, in the case of angular momenta, we need to use the rules that follow from addition of vectors). In contrast, parity is multiplicative. For a state with two particles described by wave functions $\psi_1(\mathbf{r}_1)$ and $\psi_2(\mathbf{r}_2)$ with definite parities P_1 and P_2, the parity transformation gives

$$\widehat{P}\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) = \psi_1(-\mathbf{r}_1)\psi_2(-\mathbf{r}_2) = P_1 P_2 \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2), \tag{6}$$

and so the overall parity is the product $P = P_1 P_2$.

This multiplication of parities for individual particles is useful in situations where the particles move independently in a spherical potential, such electrons in an atom or nucleons in a (nondeformed) nucleus. For example, the ground state of ^{41}Ca has even numbers of protons and neutrons filling s, p and d orbitals, plus one neutron in an f orbital. Since there is an even number of particles in each filled orbital, their parities multiply to give $+1$. The overall parity of the state is then given by that of the last neutron, which has $l = 3$ and hence $P = -1$.

We cannot use this approach for a system of two particles interacting with each other because they do not move independently. Instead we need to consider their relative and centre-of-mass motions, described by the coordinate vectors $\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$ and $\mathbf{R} = (m_1\mathbf{r_1} + m_2\mathbf{r_2})/(m_1 + m_2)$. These motions are independent and so, if both have definite orbital angular momenta, L_r and L_{cm}, we could use $(-1)^L$ to find the parity for each and then multiply these to get the overall parity.

In practice, we are interested in the angular momentum and parity of just the internal structure of a system. (The fact that the centre-of-mass motion is independent means that its momentum, orbital angular momentum and parity are separately conserved.) The wave function describing the relative motion of the particles is a function of a single vector, $\mathbf{r}$. Its parity is given in terms of the relative orbital angular momentum L_r in the usual way: $P = (-1)^{L_r}$.

Systems of three particles are more complicated in general, since there are two relative coordinate vectors. However their ground-state wave functions typically have zero orbital angular momenta and hence even parity.

Finally, in calculating the parity of a state, we need to multiply by the intrinsic parities of the particles involved. Intrinsic parity is a property of a particle, like intrinsic spin, which does not rely on it having any moving constituents. It is an extra contribution to the phase of the particle's wave function under the parity transformation. For example, a particle with intrinsic parity P_a in a state of orbital angular momentum l transforms to

$$\widehat{P}\psi_l(\mathbf{r}) = P_a\psi_l(-\mathbf{r}) = P_a(-1)^l\psi_l(\mathbf{r}), \tag{7}$$

and so has overall parity $P = P_a(-1)^l$.

The photon provides an example of this. As ultrarelativistic particles, photons require quantum field theory – the relativistic version of quantum mechanics – to describe them. In this, the polarization state of a photon is described by its electric field. This is a regular vector, transforming under parity as $\mathbf{E}(\mathbf{r}, t) \to -\mathbf{E}(-\mathbf{r}, t)$. The minus sign here indicates the photon has odd intrinsic parity, $P_\gamma = -1$.

A more subtle result of relativistic quantum mechanics is that fermions and their antiparticles must have opposite intrinsic parities. As an example, the intrinsic parities of the electron and positron satisfy

$$P_{e^-}P_{e^+} = -1. \tag{8}$$

By convention, we choose the constituents of normal matter (electrons, protons, quarks etc.) to have positive intrinsic parity, and so their antiparticles have negative parity:

$$P_{e^-} = +1, \qquad P_{e^+} = -1. \tag{9}$$

This choice means that when dealing with states of atoms or nuclei, we need to consider only the parities of the orbital wave functions of their electrons or nucleons. Note that this applies only to fermions; bosons and their antiparticles have the same parity.

In contrast, when dealing with hadrons, we need to be more careful about intrinsic parities. For example, a pion consists of a quark and an antiquark with relative $L = 0$. The opposite intrinsic parities of its constituents mean that its overall parity is odd ($P = (+1)(-1)(-1)^L = -1$).

Elementary particles, states of hadrons and states of nuclei are often labelled by J^P where J is the total internal angular momentum of the state and $P = \pm$ is the corresponding parity. (Since $|P|$ is always unity, we just need to specify the sign of P.) For example, the ground states of all even-even nuclei have $J^P = 0^+$. Note that J is often referred to as the "spin" of a state, even when the object is composite and J denotes the (vector) sum of the orbital angular momenta and intrinsic spins of all its constituents.

Returning to the example of the pion, the spins of the quark and antiquark add up to zero and so the total internal angular momentum of the state is $J = 0$. This composite particle therefore has spin and parity $J^P = 0^-$. A spin-0 object like this, which is invariant under rotations but has odd parity is known as a "pseudoscalar" to distinguish it from an ordinary scalar, which is even.

Charge conjugation

Charge conjugation describes the effect on a system of replacing every particle by its antiparticle. This can be thought of as reflecting it in a "mirror" that reverses the signs of all charges, not just electric charge but also the generalised charges of baryon number, lepton number and flavour quantum numbers. Completely neutral particles like photons and π^0 mesons, which carry none of these charges, are their own antiparticles. They remain the same under charge conjugation, except for possible changes of phase of their wave functions.

We can introduce a charge conjugation operator $\widehat{C}$. For a completely neutral state ψ, we define its charge-conjugation eigenvalue C by

$$\widehat{C}\psi = C\psi. \tag{10}$$

This eigenvalue is often referred to as the "C parity" of the state. As with ordinary parity, acting with twice $\widehat{C}$ returns the system to its original state, and so the possible values are $C = \pm 1$. Like parity, C parity is a multiplicative quantum number.

Similarly to the case of parity, completely neutral particles have intrinsic C parities. For example, the photon is its own antiparticle and it therefore has a definite C parity, corresponding to the phase change of its wave function, or rather electric field, under charge conjugation. Since charge conjugation reverses all charges of all particles in a system, including their electric charges, it reverses the sign of the electric field they produce, $\mathbf{E}(\mathbf{r}, t) \to -\mathbf{E}(\mathbf{r}, t)$. This implies that the photon has odd intrinsic C parity, $C_\gamma = -1$.

A state consisting of a particle and its antiparticle also carries no net charge of any kind and so has a definite C parity. Since charge conjugation has the effect of swapping the particle and antiparticle, the C parity can be found from the symmetry of their overall wave function. Assuming that the particles have definite orbital angular momentum L and total spin S, their wave function factorises into orbital and spin states, and we can consider the symmetry of each state separately. Swapping the positions of the particle and antiparticle reverses their relative position vector and so has the same effect as parity reversal. This contributes a factor $(-1)^L$ to the C parity of the state. The spin state of two bosons is even if their total spin is even, and odd if the total is odd. Hence swapping the spins of the particle and antiparticle contributes a factor $(-1)^S$. In contrast, the spin state of two fermions is odd if their total spin is even, and even if the total is odd. This contributes a factor of $(-1)^{S+1}$. Finally, there is also a factor of -1 which arises in quantum field theory when a fermion and its antifermion are swapped.

The bottom line is that, for both bosons and fermions, the C parity of a particle-antiparticle pair is given by

$$C = (-1)^{L+S}, \tag{11}$$

where L is their relative angular momentum and S their total spin. For example, as described above, the neutral pion consists of a superposition of $u\bar{u}$ and $d\bar{d}$ with zero orbital angular momentum and total spin. Its C parity is therefore $C_{\pi^0} = +1$.

For neutral particles and hadrons, we can extend the labelling mentioned above to J^{PC} where $C = \pm$ indicates the particle's C parity. For example, you will see the photon listed in data tables as having $J^{PC} = 1^{--}$ and the π^0 as $J^{PC} = 0^{-+}$.

Time reversal

The final discrete symmetry is time reversal, T, which reverses the flow of time: $t \to -t$. This leaves the positions of all particles unchanged but reverses their motions and spins: $\mathbf{p} \to -\mathbf{p}$, $\mathbf{J} \to -\mathbf{J}$. Unlike P and C, this transformation is not represented by a standard quantum operator since it swaps initial and final states. This means that we cannot define a corresponding eigenvalue and then make use of its conservation.

One place where time reversal does play an important role is in the CPT theorem. This is a rigorously provable result of quantum field theory (one of rather few such results; another is the connection between spin and statistics). It states that the interactions between particles should be invariant under the combination of all three discrete symmetries, C, P and T. If we take some physical process, replace all particles involved by their antiparticles, reflect it in a mirror, and then play the movie of it backwards, we get another physically allowed process. One of the consequences of this theorem is that particles and antiparticles must have identical masses. Any violation of CPT would indicate a serious flaw in our understanding of either quantum mechanics or relativity.

Conservation laws and violation of these symmetries

The symmetries of parity and charge conjugation are respected by the strong and EM interactions. As a result they are very good symmetries in much of atomic, nuclear and hadronic physics. They allow us to label states by their eigenvalues P and C. Conservation of these in strong and electromagnetic reactions and decays means that the parity and C parity of the final state of any process must match those of the initial state. We can use them to determine whether a process is allowed or not, provided we know or can deduce the relative orbital angular momenta of all the particles involved. These conservation laws, along with conservation of angular momentum, can often be used to determine the quantum numbers of new particles. In nuclear physics, similar detective work using conservation of angular momentum and P can be used to deduce the quantum numbers of excited states of nuclei.

However there is no fundamental reason why laws of physics should be invariant under these transformations individually. (After all, your image in a mirror is not a real object and indeed it differs in many little ways from the original.) As just noted, quantum mechanics and relativity require only that the combined CPT symmetry is respected.

The weak interaction provides our only current example of breaking of these discrete symmetries. Its "left-handedness" means that it violates both parity and charge conjugation as much as possible. For example, β^- decays emit right-handed antineutrinos, but never their mirror images, left-handed antineutrinos, or their charge conjugates, right-handed neutrinos.

On the other hand, β^+ decays emit only left-handed neutrinos. This implies that the combination CP is a very good symmetry of the weak interaction. Nonetheless, even this combination is broken in weak interactions, by a CP-violating phase in the couplings to the Higgs field. However this is a very small effect, which shows up only in rather special circumstances, in particular decays of neutral K and B mesons.

The CPT theorem implies that time-reversal symmetry must also be violated in a way that exactly compensates for the breaking of CP. Direct signals of this would include electric dipole moments for particles like neutrons or electrons. These are expected to be extremely small and, despite intensive efforts, they have not so far been observed.

Further reading

B. R. Martin, Nuclear and particle physics: an introduction (2nd edn.), Sec. 1.3

B. R. Martin and G. Shaw, Particle physics (3rd edn.), Secs. 5.3–5.6

D. H. Perkins, Introduction to high-energy physics (4th edn.), Secs. 3.2–3.7, 3.9–3.11

E. M. Henley and A. Garcia, Subatomic physics (3rd edn.), Ch. 9

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