PHYS 30101 APPLICATIONS OF QUANTUM PHYSICS EXAMPLES 6

These questions refer to Section 5 (lectures 19, 20 and 21).

1. (a) Show that the two-electron state with $S = 1$ and $M_S = 0$,

$$\psi_{10}(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\,\beta_z(2) + \beta_z(1)\,\alpha_z(2)\Big),$$

is an entangled state.

(b) Rewrite $\psi_{10}(1,2)$ in terms of eigenspinors of $\widehat{S}_x$,

$$\alpha_x = \frac{1}{\sqrt{2}}\Big(\alpha_z + \beta_z\Big), \qquad \beta_z = \frac{1}{\sqrt{2}}\Big(\alpha_z - \beta_z\Big),$$

and check that the state is also entangled in this basis.

(c) Is the following state entangled:

$$\phi_0(1,2) = \frac{1}{2}\Big(\alpha_z(1)\,\alpha_z(2) + \alpha_z(1)\,\beta_z(2) + \beta_z(1)\,\alpha_z(2) + \beta_z(1)\,\beta_z(2)\Big)\ ?$$

2. A quantum photocopier is a device that is intended to create an exact copy (known as a "clone") of the quantum state of some system, without altering the state of the original system. A very simple example would take two electrons, the first in some spin state $\gamma(1)$ that we want to copy, and the second in the spin-up state $\alpha(2)$. If the device works as intended, its effect should be represented by a linear operator $\widehat{X}$ that leaves both electrons in the same state:

$$\widehat{X}\,\gamma(1)\,\alpha(2) = \gamma(1)\,\gamma(2).$$

Clearly, if the first electron is either of the eigenstates of $\widehat{S}_z$, the device must be able to copy this state:

$$\widehat{X}\,\alpha(1)\,\alpha(2) = \alpha(1)\,\alpha(2), \qquad \text{and} \qquad \widehat{X}\,\beta(1)\,\alpha(2) = \beta(1)\,\beta(2).$$

Use these properties show that the device must fail when the first electron is in a general superposition of these basis states,

$$\gamma(1) = c_1\,\alpha(1) + c_2\,\beta(1), \qquad \text{with} \quad c_1, c_2 \neq 0.$$

[You should find that the two electrons are left in an entangled state instead of two copies of γ.]

3. (*From January 2012 exam paper*)

(a) A quantum device is constructed so that it can copy the circular polarisation state of a photon by emitting a second photon with the same circular polarisation. If R and L denote the right- and left-handed polarisation states, the action of the copier can be represented by

$$\widehat{X}\, R(1) = R(1)\, R(2), \qquad \text{and} \qquad \widehat{X}\, L(1) = L(1)\, L(2).$$

Show that the device fails to produce an identical copy of a photon that is in a general polarisation state. Show also that the resulting pair of photons is in an entangled state.

(b) Alice and Bob prepare pairs of photons in the entangled state

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\Big(R(1)R(2) + L(1)L(2)\Big),$$

which they use to encrypt messages by sending photon 1 to Alice and photon 2 to Bob. Each of them measures their photons with analysers that distinguish vertical and horizontal polarisations, with eigenstates

$$V = \frac{1}{\sqrt{2}}\Big(R + L\Big), \qquad \text{and} \qquad H = \frac{1}{\sqrt{2}}\Big(R - L\Big).$$

Show that Bob is able to deduce the results of Alice's measurements.

(c) Eve breaks into their communications and uses the device in part (a) to copy the polarisation of Bob's photons, leaving them with three photons in the state,

$$\widehat{X}\Psi(1,2) = \frac{1}{\sqrt{2}}\Big(R(1)R(2)R(3) + L(1)L(2)L(3)\Big),$$

where photon 3 is Eve's copy. Suppose that all three people measure their photons with analysers that distinguish vertical and horizontal polarisations, as described above. Show that Eve gets no information about the state of polarisation of Bob's photons.

(d) Explain why Alice and Bob will suspect that they have an eavesdropper.

4. One of the possible components of a classical computer is the *controlled-NOT gate*. This acts on two bits, flipping the second bit from 0 to 1 or 1 to 0 when the first has the value 1, and leaving it unchanged when the first has value 0. The first bit is known as the *control* bit since it determines whether the second is flipped. The gate leaves the control bit unaltered.

   The quantum version of this gate is one of the basic building blocks of any quantum computer. If we use the spins of two electrons to encode the Qbits, the effect of the gate can be represented by an operator $\widehat{C}$ with the properties:

$$\begin{aligned}
\widehat{C}\,\alpha(1)\alpha(2) &= \alpha(1)\beta(2), \\
\widehat{C}\,\alpha(1)\beta(2) &= \alpha(1)\alpha(2), \\
\widehat{C}\,\beta(1)\alpha(2) &= \beta(1)\alpha(2), \\
\widehat{C}\,\beta(1)\beta(2) &= \beta(1)\beta(2).
\end{aligned}$$

   Here the spin of electron 1 acts as the control Qbit.

   (a) Show that if the spin of electron 1 is in the general state

$$\gamma(1) = c_1\,\alpha(1) + c_2\,\beta(1), \qquad \text{with} \quad c_1, c_2 \neq 0$$

   and the spin of electron 2 is up, $\alpha(2)$, the gate will produce an entangled state.

   (b) The most entangled states possible for a pair of spin-1/2 particles are the "Bell states":[1]

$$\begin{aligned}
\psi_1(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1)\,\beta(2) - \beta(1)\,\alpha(2)\Big), \\
\psi_2(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1)\,\beta(2) + \beta(1)\,\alpha(2)\Big), \\
\psi_3(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1)\,\alpha(2) - \beta(1)\,\beta(2)\Big), \\
\psi_4(1,2) &= \frac{1}{\sqrt{2}}\Big(\alpha(1)\,\alpha(2) + \beta(1)\,\beta(2)\Big).
\end{aligned}$$

   Show that the cNOT gate converts each of these into an unentangled state.

   (c) Show also that we can measure the Bell state of a pair of electrons by first applying the cNOT gate to the pair and then measuring S_x of electron 1 and S_z of electron 2.

[1] These states have $S = 0$ or $S = 1$ and $S_z = 0$ or $S_x = 0$ or $S_y = 0$. They are named after John Bell, the Northern Irish physicist who proved a number of important results for the theory of quantum measurement.

5. [*Challenge question*] Alice has an electron that she has carefully prepared in the spin state

$$\gamma(1) = c_1\,\alpha(1) + c_2\,\beta(1).$$

She wishes to send this to her colleague Bob at the other side of the campus. Paranoid as ever, the twins are unwilling to entrust their precious state to the internal mail. Instead, Bob prepares a pair of electrons in a state with total spin zero,

$$\psi_{00}(2,3) = \frac{1}{\sqrt{2}}\Big(\alpha(2)\,\beta(3) - \beta(2)\,\alpha(3)\Big).$$

He then sends electron 2 to Alice. (If it gets lost in the mail, he can easily create another entangled pair and try again.) Alice takes her electron and the one she has just received from Bob and feeds them into the device in the previous question that measures the Bell state of this pair of electrons. She then phones Bob to give him the (classical) information on the result of her measurement.

The full state of the three electrons before Alice's measurement is

$$\Psi(1,2,3) = \gamma(1)\,\psi_{00}(2,3).$$

Expand this in terms of the four Bell states $\psi_{1,2,3,4}(1,2)$ for electrons 1 and 2. [*Hint:* Add and subtract pairs of the $\psi_i(1,2)$ to get expressions for $\alpha(1)\,\alpha(2)$ etc. Then multiply out $\Psi(1,2,3)$ and use your results for the products of basis states of electrons 1 and 2.]

Use this expansion to show that, when Bob is given the result of Alice's measurement, he knows exactly what the state of his electron (3) is, without making any measurements on it. Show also that this state is either $\gamma(3)$ or a very closely related state involving the same coefficients $c_{1,2}$.