PHYS 30101 APPLICATIONS OF QUANTUM PHYSICS EXAMPLES 5

These questions refer to Sections 3 (lecture 15) and 4 (lectures 16 to 18).

1. The lowest two states of an electron in a double quantum dot are described by the 2×2 matrix Hamiltonian

$$\widehat{H} = \frac{1}{2}\begin{pmatrix} \epsilon & -\Delta \\ -\Delta & -\epsilon \end{pmatrix},$$

where ϵ is the difference in potential energies between the two dots, and Δ the coupling between them (arising from tunnelling). The basis vectors

$$L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describe states where the electron is trapped on the left- and right-hand dot respectively.

   (a) Find the energy eigenvalues of this system.

   (b) Explain why, if $\Delta \ll \epsilon$, the energy eigenvectors are well approximated by L and R. Use first-order perturbation theory to show that their energies are $\pm\epsilon/2$ to first order in Δ.

   (c) Find the energy eigenvectors for the case $\epsilon = 0$. Use these to construct a general solution $\psi(t)$ to the TDSE in this case.

   (d) If the electron is initially located on the left-hand dot, $\psi(0) = L$, and $\epsilon = 0$, obtain the probability of finding it on the right-hand dot at a later time t.

2. The $3d$ ($n = 3$, $l = 2$) level in hydrogen is split by the spin-orbit interaction into two levels with an energy difference of 4.5×10^{-6} eV. State the values of the quantum number j for these levels, indicating which has the lower energy, and give their degeneracies.

   The spin-orbit interaction has the form

$$\widehat{H}_{so} = f(r)\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}}.$$

   Use the observed splitting to determine the radial matrix element

$$\mathcal{E}_{nl} = \langle f(r) \rangle \hbar^2$$

   for the $3d$ level. Show that the average energy of the set of $3d$ states is unchanged by this interaction.

3. The first excited levels of calcium have the outer electrons in a $(4s)^1\,(4p)^1$ configuration. Three of these are seen, with energies 1.87908 eV, 1.88555 eV and 1.89868 eV. Assuming that they are split by the spin-orbit interaction,

$$\widehat{H}_{so} = \frac{\mathcal{E}}{\hbar^2}\,\widehat{\mathbf{L}}\cdot\widehat{\mathbf{S}},$$

deduce the angular-momentum quantum numbers (L, S and J) of these levels. Compare the ratio of their splittings with the prediction of first-order perturbation theory.

4. Sketch the energy-level diagram for the $3d$ states of hydrogen in a weak magnetic field, indicating the quantum numbers of the levels. Calculate the magnetic splittings for a field of 10^{-2} T. [You may use the value for the Bohr magneton, $\mu_B = e\hbar/2m_e = 5.8\times10^{-5}$ eV/T.]

5. Find the Landé g-factors for the levels of calcium in question 3. Sketch a similar diagram for the splittings of these levels in a weak magnetic field. Estimate the size of the magnetic field for which the weak-field approximation breaks down.

6. The orbital motion of an electron in a thin cylindrical quantum dot is governed by an electrostatic potential that is well approximated by a two-dimensional harmonic oscillator. It can be described by the Hamiltonian

$$\widehat{H}_{\rm orb} = -\frac{\hbar^2}{2m}\,\nabla^2 + \frac{1}{2}\,k\,(x^2+y^2),$$

where, for a thin enough dot, we may neglect any excitation in the z direction (see Section 2.2 in lecture 6). Show that the energy eigenvalues of the electron have the form

$$E_{\rm orb} = (N+1)\hbar\,\omega, \qquad \text{where } N = 0, 1, 2, \ldots,$$

and find the degeneracies of the first four levels.

After some heavy lifting with Hermite polynomials, it is possible to form linear combinations of the states in a given level that have definite values for L_z. The corresponding quantum number m_l can be shown to run over even numbers from $-N$ to $+N$ if N is even, and odd numbers from $-N$ to $+N$ if it is odd. Check that the degeneracies that you found above are consistent with this pattern.

As in a real atom, the electrostatic potential leads to a (small) spin-orbit interaction. If the dot is placed in a magnetic field B along its cylindrical axis, the full Hamiltonian for the electron is

$$\widehat{H} = \widehat{H}_{\rm orb} + \frac{\mathcal{E}_{\rm so}}{\hbar^2}\,\widehat{L}_z\widehat{S}_z + \frac{e}{2m}\left(\widehat{L}_z + 2\,\widehat{S}_z\right)B.$$

The states in the energy levels of the oscillator will be split by this depending on their quantum numbers m_l and m_s. Sketch, as functions of B, the energies of the states in the first two oscillator levels ($N = 0$ and 1).