PHYS 30101 APPLICATIONS OF QUANTUM PHYSICS EXAMPLES 4

These questions refer to Section 3 (lectures 11 to 14).

1. (a) Use the commutation rules for the angular momentum operators to show that

$$\widehat{L}_+\widehat{L}_- = \widehat{\mathbf{L}}^2 - \widehat{L}_z^2 + \hbar\widehat{L}_z.$$

   (b) A particle is in a state $\phi_{l\,m}$, with definite values for $|\mathbf{L}|^2$ and L_z. By writing $\widehat{L}_x$ in terms of $\widehat{L}_+$ and $\widehat{L}_-$, show that $\langle\widehat{L}_x\rangle$ is zero for this state.
   [If you are happy with Dirac notation, you can denote the state by $|l\,m\rangle$.]

2. Evaluate the products

$$\sigma_2\,\sigma_3 \quad \text{and} \quad \sigma_3\,\sigma_2,$$

   where $\sigma_{1,2,3}$ are the Pauli matrices. Hence find the commutator

$$[\sigma_2, \sigma_3],$$

   and show that this leads to the usual angular-momentum commutation relation for the operators $\widehat{S}_y$ and $\widehat{S}_z$.

3. (a) Find normalised eigenvectors α_y, β_y of the operator $\widehat{S}_y = (\hbar/2)\sigma_2$ for a particle with spin-1/2.

   (b) The state of a spin-1/2 particle is given by the normalised spinor

$$\gamma = \frac{1}{\sqrt{5}}\begin{pmatrix} \mathrm{i} \\ 2 \end{pmatrix}.$$

   Evaluate the expectation value of $\widehat{S}_y$ in this state.

   (c) Write γ as a linear combination of the spinors α_y, β_y that you got in part (a). [*Hint: You may find the orthogonality of eigenvectors useful here.*] Hence find the probabilities of getting $+\hbar/2$ or $-\hbar/2$ in a measurement of S_y. Check that these are consistent with your result for part (b).

4. Calculate the energy splitting (in eV) between the two spin states of an electron in: (a) the Earth's magnetic field, $B \simeq 50\ \mu$T; (b) the field of the magnet that Andre Geim (our Ignobel as well as Nobel prize winner) used to levitate a frog, $B = 16$ T; (c) the field near the surface of a typical young pulsar, $B \simeq 10^8$ T. Comment on where in the electromagnetic spectrum, we might observe radiation from transitions between the spin states in each of these cases. [You may find the value for the Bohr magneton in atomic units useful: $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.]

5. An electron is in a magnetic field that lies in the xz plane,

$$\mathbf{B} = B_x\mathbf{i} + B_z\mathbf{k}.$$

Write down the Hamiltonian operator for the interaction of the electron's intrinsic magnetic moment with this field and express it in matrix form. Find its eigenvalues and sketch these as a function of B_z, for fixed, nonzero B_x. How would the picture differ if B_x were zero?

6. An electron sits in a magnetic field B that points in the $+y$ direction. The Hamiltonian describing the interaction of its spin with the field is

$$\widehat{H} = \frac{eB}{m}\,\widehat{S}_y.$$

The spinor $\chi(t)$ describing the state of of the electron evolves according to the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\,\frac{\partial\chi}{\partial t} = \widehat{H}\chi.$$

(a) Using the spinors α_y, β_y you found in question 3(a), write down two separable solutions of this equation. Use these to construct a general time-dependent solution.

(b) At time $t = 0$ the electron is in the state γ introduced in question 2(b): $\chi(0) = \gamma$. Find the state of the electron as a function of t.

(c) How does the probability of measuring $S_y = +\hbar/2$ depend on time?

(d) How does the probability of getting $S_z = +\hbar/2$ depend on time?

7. Two electrons have total angular-momentum quantum numbers of $j_1 = 3/2$ and $j_2 = 5/2$. List the allowed values of J and M_J (the quantum numbers that give the eigenvalues of the total angular momentum operators $\widehat{\mathbf{J}}^2 = (\widehat{\mathbf{J}}^{(1)} + \widehat{\mathbf{J}}^{(2)})^2$ and $\widehat{J}_z = \widehat{J}_z^{(1)} + \widehat{J}_z^{(2)}$). Check that the number of eigenstates of these operators agrees with the total number of states of the two electrons with definite values of both $J_z^{(1)}$ and $J_z^{(2)}$.

8. An electron has orbital and spin angular-momentum quantum numbers $l = 1$ and $s = 1/2$, respectively. The raising and lowering operators for the total angular momentum are

$$\widehat{J}_+ = \widehat{L}_+ + \widehat{S}_+, \qquad \widehat{J}_- = \widehat{L}_- + \widehat{S}_-,$$

where

$$\widehat{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Acting on normalised angular-momentum eigenfunctions $\phi_{l\,m}(\mathbf{r})$ with $\widehat{L}_\pm$ gives

$$\begin{aligned} \widehat{L}_+ \phi_{l\,m} &= \sqrt{l(l+1) - m(m+1)}\,\hbar\,\phi_{l\,m+1} \\ \widehat{L}_- \phi_{l\,m} &= \sqrt{l(l+1) - m(m-1)}\,\hbar\,\phi_{l\,m-1}. \end{aligned}$$

(a) Start with the state with the largest possible value of $m_j = m_l + m_s$,

$$\psi_{+3/2}(\mathbf{r}) = \phi_{1\,+1}(\mathbf{r})\alpha.$$

What do you get when you act on this state with $\widehat{J}_+$? What does this tell you about the quantum number j for this state?

(b) Act on $\psi_{+3/2}$ with $\widehat{J}_-$ to get a new state (which should take the form of a superposition of $\phi_{1\,0}\,\alpha$ and $\phi_{1\,+1}\,\beta$). What are the quantum numbers j and m_j for this state?

(c) A second state ψ' can be formed as a superposition of $\phi_{1\,0}\,\alpha$ and $\phi_{1\,+1}\,\beta$. It satisfies $\widehat{J}_+\psi' = 0$. Without explicitly constructing this state, deduce its quantum numbers j and m_j.

(d) By acting repeatedly on each of these states with $\widehat{J}_-$ we could form two "ladders" of eigenstates of $\widehat{\mathbf{J}}^2$. How many states are there in each ladder? Are there any other eigenstates of $\widehat{\mathbf{J}}^2$ that could be constructed out of states with $l = 1$ and $s = 1/2$?