PHYS 30101 APPLICATIONS OF QUANTUM PHYSICS EXAMPLES 3

These questions refer to Section 2 (lectures 6 to 10).

1. A quantum dot consists of a thin square slice of undoped semiconductor, with sides of length a. The potential energy of an electron in the dot is zero. Assuming that the potential can be regarded as infinite at the edges of the square, find the first three energy eigenvalues of a single electron in the dot. (You can assume that the thickness of the dot is much smaller than a.) What are the degeneracies of these levels? What numbers of electrons correspond to the first three closed shells?

   The dot is placed between two electrical contacts (the drain and source) and is separated from them by thin insulating layers. It is also surrounded by a gate, which can be used to change the constant potential inside the dot. The drain and source are held at the same voltage. Explain briefly why the dot will conduct current only for certain discrete values of the gate voltage. Describe the pattern you expect for these values.

2. The first three energy eigenfunctions of a one-dimensional harmonic oscillator are

$$\begin{aligned}\psi_0(x) &= N_0 \exp\left(-\frac{x^2}{2b^2}\right),\\ \psi_1(x) &= N_1\, x \exp\left(-\frac{x^2}{2b^2}\right),\\ \psi_2(x) &= N_2 \left(\frac{x^2}{b^2} - \frac{1}{2}\right) \exp\left(-\frac{x^2}{2b^2}\right),\end{aligned}$$

   where $b = \sqrt{\hbar/m\omega}$ and the N_n are real normalisation constants.

   (a) Determine the normalisation constants for these three functions.
   [You may use the integrals

$$\begin{aligned}\int_{-\infty}^{+\infty} \exp(-y^2)\,\mathrm{d}y &= \sqrt{\pi},\\ \int_{-\infty}^{+\infty} y^2 \exp(-y^2)\,\mathrm{d}y &= \frac{\sqrt{\pi}}{2},\\ \int_{-\infty}^{+\infty} y^4 \exp(-y^2)\,\mathrm{d}y &= \frac{3\sqrt{\pi}}{4},\end{aligned}$$

   and you should not need to evaluate any integrals with odd powers of x.]

   (b) Show that these three functions are orthogonal, that is they satisfy:

$$\int_{-\infty}^{+\infty} \psi_n^*(x)\psi_m(x)\,\mathrm{d}x = 0, \qquad \text{if } n \neq m.$$

   Sketch the three functions and use your pictures to explain this mathematical result.

(c) Using orthogonality (or otherwise) express the normalised function

$$\phi(x) = \frac{2}{\left(3\sqrt{\pi}\, b^5\right)^{1/2}}\, x^2 \exp\left(-\frac{x^2}{2b^2}\right)$$

as a superposition of these energy eigenfunctions. Hence find the probabilities that the energy of a particle in the state $\phi(x)$ is measured to be $\frac{1}{2}\hbar\omega$, $\frac{3}{2}\hbar\omega$ or $\frac{5}{2}\hbar\omega$. Could a measurement of the energy of a particle in this state give any other values?

3. An electron is trapped in a thin layer of semiconductor between two insulators. The insulators can be regarded as providing infinite potential walls at $x = 0$ and $x = a$. If we treat the potential in the semiconductor as zero, this becomes the usual infinite square well. Write down the energy and the normalised wave function for the ground state of the electron. (Treat this as a one-dimensional problem.)

   More realistically, the potential inside the semiconductor is given by

$$V(x) = \frac{1}{2} K \left(x - \frac{a}{2}\right)^2 \quad \text{for } 0 < x < a.$$

   We can continue to treat the potential outside the well as infinite. Assuming that the constant K is small, use first-order perturbation theory to calculate the ground-state energy of the electron. [*Hint: you may want to use a double-angle formula, followed by some integrating by parts.*]

4. A quantum well consists of a slab of conducting material where electrons are trapped by a harmonic oscillator potential,

$$V(z) = \frac{1}{2} m^* \omega^2 z^2,$$

   in the z direction, but are free to move in the xy plane. Consider a large square region of this, with sides of length L in the x and y directions, and assume that the wave functions vanish on the edges of this square. Write down the energy eigenvalues of an electron in this well.

   Find the density of states, $\mathrm{d}n/\mathrm{d}k$, as a function of $k = \sqrt{k_x^2 + k_y^2}$, the magnitude of the wave vector in the xy plane.

   Rewrite this as the density of states as a function of energy, $\mathrm{d}n/\mathrm{d}E$, for a particular eigenvalue of the trapped motion.

   Finally, sketch the total density of states as a function of energy E. (You will need to add up the contributions from all the eigenvalues for the z motion that are less than E and so the result will not be a smooth function.)