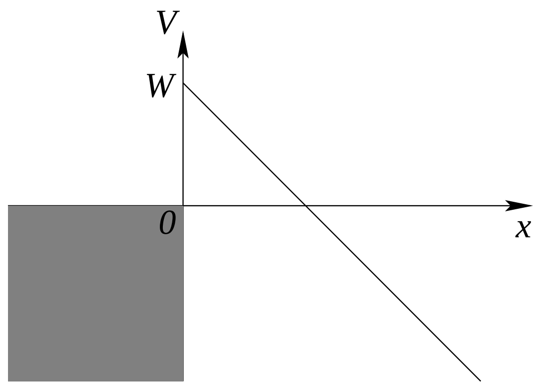
Questions 1–3 refer to Section 1 Tunnelling (lectures 2–4).
Questions 4–5 refer to Section 2 Trapped particles (lectures 5 and 6).

1. What is the tunnelling factor for an electron to get through a square barrier of width 1.6 nm, if the energy of the electron is 1 eV below the top of the barrier?

2. Cold emission occurs when electrons tunnel out from the surface of a conductor that has a very strong electric field outside it.[1] Near the surface, the potential energy of an electron has the form

$$V(x) = \begin{cases} W - e\mathcal{E}x & \text{for} \quad x > 0 \\ -V_0 & \text{for} \quad x < 0 \end{cases}.$$

This is shown in the figure. Here W is the work function (the minimum energy needed to remove an electron from the conductor, as in the photoelectric effect) and $\mathcal{E}$ is the electric field outside the metal.



[The shaded region shows the "Fermi sea" of levels occupied by electrons inside the metal. Here we only need to consider states at the top of the sea, which have the shortest distance to tunnel through.]

Show that the tunnelling factor for an electron with zero energy is

$$T = \exp\left[-\frac{4\sqrt{2m}}{3\hbar e\mathcal{E}}\, W^{3/2}\right].$$

[*Hints: first find the limits of the classically forbidden zone. Then, to do the integral, you may find it helpful to change variables to* $y = W - e\mathcal{E}x$).]

The work function for a typical metal is 4 eV. Evaluate the tunneling factor for an electric field of 3×10^9 V/m. How would this change if the surface were coated with caesium, which has a work function of about 2 eV?

[1] This is used in the backlights for LCD displays in computer monitors or TVs, and in some low-energy lightbulbs. It is also how data on your memory stick is erased: when an electric field is applied electrons tunnel across a thin insulating layer out of the memory cells. The fields near nanometer-sized surface irregularities or across a 10 nm insulating layer can be huge, $\sim 10^9$ V/m.

3. In lectures we looked at resonant tunnelling through a pair of square barriers. We saw that if the energy exactly matched that of a resonance, there was perfect constructive interference between all the waves reflected between the walls and 100% of the incoming wave is transmitted through the pair of barriers (just like the situation at a bright fringe of an etalon).

   Without doing any detailed calculations, describe the forms of the wave functions in the regions to the left of the barriers, between the barriers, and to the right of both barriers, for an incoming wave from the left

   (a) at an energy that is only slightly different from one of the resonant energies,

   (b) at an energy that lies in-between two resonances.

   [Assume that the barriers are wide and the energies are well below the top of the barriers.]

4. An electron is trapped inside a sphere of radius R by a potential that we can treat as infinite for $r > R$. Inside the sphere, the electron experiences no potential and so its Hamiltonian is

$$\widehat{H} = -\frac{\hbar^2}{2m^*}\nabla^2.$$

   In spherical polar coordinates (r, θ, ϕ), we can express ∇^2 in the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\widehat{\mathbf{L}}^2}{r^2},$$

   where $\widehat{\mathbf{L}}^2$ is the square of the angular momentum operator.

   (a) Show that the spherically symmetric wave function

$$\psi(r) = \frac{u(r)}{r}$$

   can be a solution of the TISE provided $u(r)$ satisfies

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = -k^2\, u(r), \qquad \text{where} \quad k^2 = \frac{2m^* E}{\hbar^2}.$$

   (b) State the boundary conditions on $u(r)$, explaining how they relate to the conditions on $\psi(r)$. Hence show the energy eigenvalues for the spherically symmetric states of an electron in this sphere are

$$E_n = \frac{\hbar^2}{2m^*}\left(\frac{n\pi}{R}\right)^2, \qquad \text{where} \quad n = 1, 2, 3, \ldots.$$

   (c) Electrons in solid ZnS have an effective mass $m^* = 0.2m_e$. Calculate the energies of the lowest three spherical states of an electron in a ZnS sphere of radius 2 nm. Where in the spectrum would you expect to see radiation from transitions between these states?

5. A simple model for a nucleus (the "shell model") consists of protons and neutrons moving in a three-dimensional harmonic-oscillator potential,

$$V(x, y, z) = \frac{1}{2} k (x^2 + y^2 + z^2).$$

Use the fact that the Hamiltonian can be separated in Cartesian coordinates to write down the energy eigenvalues of a nucleon (a proton or a neutron) in this potential. Find also the degeneracies of the first three levels.

Nucleons are spin-1/2 particles, like electrons. Find the "magic numbers" that correspond to nuclei with closed shells of either protons or neutrons. Use these to explain why the nuclei ^{4}He, ^{16}O and ^{40}Ca are commonly found on Earth and elsewhere in the Universe.

The rest energy of a nucleon is approximately $Mc^2 = 940$ MeV. In a typical nucleus, the energy needed to excite a nucleon from one shell to the next is $\hbar\omega = 10$ MeV. Find, in fm, the corresponding oscillator length parameter,

$$b = \sqrt{\frac{\hbar}{M\omega}}.$$

Comment on the size of your result. [*Hint: you may find it helpful to use the fact that* $\hbar c = 200$ MeV fm *in units appropriate to nuclear physics.*]