Lecture 11

General properties of angular momenta

Commutation rules

$$\left[\widehat{L}_x, \widehat{L}_y\right] = \mathrm{i}\,\hbar\,\widehat{L}_z \qquad \text{and cycle } x, y, z$$

$\longrightarrow$ components of **L** are incompatible observables

$\longrightarrow$ at best, know $|\mathbf{L}|^2$ and one component

$$\widehat{\mathbf{L}}^2\phi_{LM} = L(L+1)\,\hbar^2\,\phi_{LM} \qquad \widehat{L}_z\,\phi_{LM} = M\,\hbar\,\phi_{LM}$$

Raising and lowering operators

$$\widehat{L}_+ = \widehat{L}_x + \mathrm{i}\,\widehat{L}_y \qquad \widehat{L}_- = \widehat{L}_x - \mathrm{i}\,\widehat{L}_y$$

act on ϕ_{LM} to produce new eigenstates with same $|\mathbf{L}|^2$
and with M stepped up or down by 1

$$\begin{aligned}
\widehat{L}_z\,\widehat{L}_+\,\phi_{LM} &= (M+1)\hbar\,\widehat{L}_+\,\phi_{LM} \\
\widehat{L}_z\,\widehat{L}_-\,\phi_{LM} &= (M-1)\hbar\,\widehat{L}_-\,\phi_{LM}
\end{aligned}$$

Values of M run from $-L$ to $+L$ in steps of $1 \longrightarrow 2\,L$ must be integer

Orbital wave functions are periodic $\longrightarrow M$, L are integers ($L \geq 0$)

Eigenvalues of $\widehat{\mathbf{L}}^2$, $\widehat{L}_z$ are

$$|\mathbf{L}|^2 = L(L+1)\,\hbar^2 \qquad \text{where } L = 0, 1, 2, \cdots$$

$$L_z = M\,\hbar \qquad \text{where } M = -L, -L+1, \cdots, +L-1, +L$$

($2L+1$ values of M for given L)

But half-integer quantum numbers are also mathematically consistent