In this paper α_z and β_z denote the eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the value for the Bohr magneton, $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.

1. (a) Estimate the probability for an α particle to tunnel through a square barrier 10 fm wide when the energy of the particle is 10 MeV below the top of the barrier. [5 marks]

   (b) The electrons in an excited atom have a total orbital angular-momentum quantum number $L = 1$ and a total spin quantum number $S = 2$. What are the possible values of J and M_J, the quantum numbers describing the total angular momentum? [5 marks]

   (c) For the atom in part (b), find the splitting of the level with the lowest value of J in a magnetic field of 0.5 T. [5 marks]

   (d) Show that

   $$\alpha_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

   is an eigenstate of the spin operator along the y axis, $\widehat{S}_y$, and find its eigenvalue. Show also that α_y is not an eigenstate of the spin along the x axis, $\widehat{S}_x$. [5 marks]

   (e) The spin of an electron is described by the normalised spinor

   $$\gamma = \frac{1}{2} \begin{pmatrix} 1 \\ \mathrm{i}\sqrt{3} \end{pmatrix}.$$

   Find the probability that a measurement of the spin along the y-axis gives the result $+\hbar/2$. [5 marks]

2. A quantum dot consists of a thin square block of undoped InGaAs, with sides of length a and a thickness of $a/5$. It is surrounded by insulating walls, where the potential energy of the electron can be assumed to be infinite. The potential energy of an electron inside the dot is zero.

   (a) Give expressions for the first three energy levels of a single electron in the dot. What are the degeneracies of these levels? What "magic" numbers of electrons correspond to the first three closed shells? [10 marks]

   (b) Find the energy needed to excite an electron from its ground state to its first excited state for a block with $a = 30$ nm, taking the effective mass of an electron in InGaAs to be $m^* = 0.04\, m_e$. Where in the electromagnetic spectrum would you expect to see radiation from transitions between states in this dot? [4 marks]

   (c) The dot is placed between two electrical contacts (the drain and source) which are parallel to its square faces and separated from them by thin insulating layers. It is also surrounded by a gate, which can be used to change the constant potential inside the dot. The drain and source are held at the same voltage. Explain why the dot will conduct current only for certain discrete values of the gate voltage. Describe the pattern you expect for the spacings between these values. [11 marks]

3. (a) A system is described by the Hamiltonian

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_1,$$

where the eigenvalues and eigenfunctions of $\widehat{H}_0$ are known and $\widehat{H}_1$ can be treated as a small perturbation. Write down an expression for the eigenvalues of $\widehat{H}$ to first order in $\widehat{H}_1$. State any assumptions you have made about the eigenfunctions of $\widehat{H}_0$. [4 marks]

(b) Explain briefly the origin of the spin-orbit interaction in a one-electron atom, commenting on the consquences of the sign of the interaction. [5 marks]

(c) The spin-orbit interaction energy operator has the form

$$\widehat{H}_{\mathrm{so}} = A\,\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}}.$$

Rewrite this in a form that shows that eigenstates of the total angular momentum operator $\widehat{\mathbf{J}}^2$ are not mixed by this interaction. Expain briefly why this form is useful, relating this to your answer to part (a). [7 marks]

(d) The spectrum of atomic sodium contains a pair of intense yellow lines corresponding to the transitions ${}^2P_{3/2} \rightarrow {}^2S_{1/2}$ at 589.0 nm and ${}^2P_{1/2} \rightarrow {}^2S_{1/2}$ at 589.6 nm. Assuming that the splitting is due to the spin-orbit interaction as given in part (c), find the value in eV of the constant $A\hbar^2$. [9 marks]

4. One of the basic components of a quantum computer is the *controlled-NOT gate*. This acts on two Qbits, flipping the state of the second Qbit if the first has the value 1, and leaving it unchanged if the first has value 0. If the spins of two electrons are used to encode the Qbits, the effect of the gate can be represented by the operator $\widehat{C}$ with the properties:

$$\begin{aligned}
\widehat{C}\,\alpha_z(1)\alpha_z(2) &= \alpha_z(1)\beta_z(2), \\
\widehat{C}\,\alpha_z(1)\beta_z(2) &= \alpha_z(1)\alpha_z(2), \\
\widehat{C}\,\beta_z(1)\alpha_z(2) &= \beta_z(1)\alpha_z(2), \\
\widehat{C}\,\beta_z(1)\beta_z(2) &= \beta_z(1)\beta_z(2).
\end{aligned}$$

Here the spin of electron 1 is known as the *control* Qbit since it determines whether the spin of electron 2 is flipped.

(a) Consider the initial state where electron 1 has spin-up along the x axis and electron 2 has spin-up along the z axis:

$$\alpha_x(1)\,\alpha_z(2).$$

Show that the controlled-NOT gate acts on this state to produce an entangled state. [You may use the expressions for the eigenstates of spin along the x axis:

$$\alpha_x = \frac{1}{\sqrt{2}}\Big(\alpha_z + \beta_z\Big), \qquad \beta_x = \frac{1}{\sqrt{2}}\Big(\alpha_z - \beta_z\Big).\ ]$$

[6 marks]

(b) Show that when both electrons are initially spin-up along the x axis,

$$\alpha_x(1)\,\alpha_x(2),$$

the gate produces an unentangled state. [7 marks]

(c) By finding the effect of the gate on all the states where the electrons have spins either up or down along the x axis,

$$\alpha_x(1)\,\alpha_x(2), \quad \alpha_x(1)\,\beta_x(2), \quad \beta_x(1)\,\alpha_x(2), \quad \beta_x(1)\,\beta_x(2),$$

show that for these states, it acts as a controlled-NOT gate with the spin of electron 2 as the control Qbit. [12 marks]

NUMERICAL AND BOTTOM-LINE ANSWERS

1. (a) $T \sim 10^{-12}$ (depending on the accuracy of your input numbers)

   (b) $J = 1, 2, 3$; $M_J = -3, \ldots, +3$

   (c) 3 equally spaced levels with splitting $\Delta E = 7.3 \times 10^{-5}$ eV

   (d) No numerical answer

   (e) $P = \frac{1}{4}\left(2 + \sqrt{3}\right)$

2. (a) Lowest eigenvalues:

$$
\begin{aligned}
E_{111} &= 27\,\frac{\hbar^2\,\pi^2}{2\,m^* a^2} \\
E_{121} &= 30\,\frac{\hbar^2\,\pi^2}{2\,m^* a^2} \\
E_{221} &= 33\,\frac{\hbar^2\,\pi^2}{2\,m^* a^2}
\end{aligned}
$$

   with degeneracies 1, 2, 1 (not counting spin states)

   Corresponding magic numbers: 2, 6, 8

   (b) $E_{111} \simeq 8 \times 10^{-2}$ eV
   Infrared radiation

   (c) No numerical answer

3. (a) No numerical answer

   (b) No numerical answer

   (c) $A\hbar^2 = 1.4 \times 10^{-3}$ eV

4. (a) $\widehat{C}\,\alpha_x(1)\,\alpha_z(2) = \dfrac{1}{\sqrt{2}}\Big[\alpha_z(1)\,\beta_z(2) + \beta_z(1)\,\alpha_z(2)\Big]$

   (b) $\widehat{C}\,\alpha_x(1)\,\alpha_x(2) = \alpha_x(1)\,\alpha_x(2)$

   (c)

$$
\begin{aligned}
\widehat{C}\,\alpha_x(1)\,\alpha_x(2) &= \alpha_x(1)\,\alpha_x(2) \\
\widehat{C}\,\alpha_x(1)\,\beta_x(2) &= -\beta_x(1)\,\beta_x(2) \\
\widehat{C}\,\beta_x(1)\,\alpha_x(2) &= \beta_x(1)\,\alpha_x(2) \\
\widehat{C}\,\beta_x(1)\,\beta_x(2) &= -\alpha_x(1)\,\beta_x(2)
\end{aligned}
$$