In this paper α_z and β_z denote the eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the value for the Bohr magneton, $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.

1. (a) An electron is trapped inside a cuboidal quantum dot with sides a, a and $a\sqrt{3}$. You may assume that the potential energy outside the dot is very large. Write down an expression for the energy levels of the electron, stating clearly the ranges of the quantum numbers. [5 marks]

   (b) Describe briefly what is meant by "Coulomb blockade" in a quantum dot. [5 marks]

   (c) Earlier this year, researchers produced a record magnetic field of 100.75 T. Find the splitting (in eV) of the ${}^2P_{1/2}$ level of rubidium in this field. (You may assume this splitting is smaller than the spin-orbit splitting in this atom.) [5 marks]

   (d) The spin-orbit interaction energy of an electron in an atom has the form

   $$\widehat{H}_{so} = A\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}},$$

   where A is a constant. Show that this can be rewritten in terms of $\widehat{\mathbf{L}}^2$, $\widehat{\mathbf{S}}^2$ and $\widehat{\mathbf{J}}^2$. Hence write down an expression for the eigenvalues of $\widehat{H}_{so}$ in terms of the corresponding angular-momentum quantum numbers. [5 marks]

   (e) The spins of two electrons are in the state

   $$\psi(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\,\beta_z(2) + \beta_z(1)\,\alpha_z(2)\Big).$$

   Why do we describe this as an "entangled" state? [5 marks]

2. (a) Explain briefly what is meant by "tunnelling" in quantum mechanics. Give an example of its importance for nanoscale electronics. [4 marks]

(b) Without any detailed algebra, explain the origin of the formula

$$T \simeq \exp\left[-2\int_a^b \sqrt{\frac{2m}{\hbar^2}\Big(V(x)-E\Big)}\,\mathrm{d}x\right]$$

for the tunneling factor for a particle of energy E to tunnel through a barrier $V(x)$. Your answer should state briefly the conditions under which this is a good approximation, and define the points a, b. [8 marks]

(c) In an insulating layer between two electrical contacts, the potential energy of an electron can be approximated by a parabolic barrier:

$$V(x) = V_0 - \frac{1}{2}\,k\,x^2,$$

where V_0 is the maximum height of the potential and k is a constant. An electron is incident on this barrier with an energy $E < V_0$. Show that the classically forbidden region for the electron has a length $2b$ where

$$\frac{1}{2}\,k\,b^2 = V_0 - E.$$

Use the formula in part (b) to show that the tunnelling factor for this electron is

$$T \simeq \exp\left[-\pi\sqrt{\frac{mk}{\hbar^2}}\,b^2\right].$$

[7 marks]

[You may use the integral,

$$\int_{-1}^{+1} \sqrt{1-y^2}\,\mathrm{d}y = \frac{\pi}{2}\,.\,]$$

(d) Show that the tunnelling factor in part (c) can be rewritten in the form

$$T \simeq \exp\left[-\pi\,b\sqrt{\frac{2m}{\hbar^2}(V_0-E)}\right].$$

Hence estimate the tunnelling factor for an electron in an insulating layer where the maximum height of the potential is $V_0 - E = 1$ eV and the width of the forbidden zone is $2b = 3$ nm. Calculate the tunnelling factor for a rectangular barrier of the same height and width, and comment on your results. [6 marks]

3. (a) Angular momentum operators satisfy the commutation rules

$$\left[\widehat{L}_x, \widehat{L}_y\right] = i\hbar\widehat{L}_z, \quad \left[\widehat{\mathbf{L}}^2, \widehat{L}_x\right] = 0, \quad \text{etc.}$$

Decribe briefly what these imply for eigenstates of angular momentum.

[3 marks]

(b) Using the commutation rules given in part (a), show that the operator $\widehat{L}_{\rm up} = \widehat{L}_y - i\widehat{L}_x$ satisfies

$$\left[\widehat{L}_z, \widehat{L}_{\rm up}\right] = \hbar\widehat{L}_{\rm up}, \quad \left[\widehat{\mathbf{L}}^2, \widehat{L}_{\rm up}\right] = 0.$$

Hence show that $\widehat{L}_{\rm up}$ acts on an eigenstate ϕ_{lm} of $\widehat{L}_z$ and $\widehat{\mathbf{L}}^2$ to produce another eigenstate with the same value of l and with m raised by one. [8 marks]

(c) The two outer electrons in a barium atom are excited into a state where one has an orbital angular momentum quantum number $l_1 = 2$ and the other $l_2 = 3$. The total spin of the electrons is zero.

i. How many possible states are there with definite values of $L_z^{(1)}$ and $L_z^{(2)}$ for each of the two electrons? (Here $\widehat{\mathbf{L}}^{(i)}$ denotes the operator acting on electron i.) [3 marks]

ii. Magnetic interactions between the electrons couple their angular momenta to form eigenstates of total orbital angular momentum $\widehat{\mathbf{L}}^2 = \left(\widehat{\mathbf{L}}^{(1)} + \widehat{\mathbf{L}}^{(2)}\right)^2$. List the possible values for the corresponding quantum number L. Show that the total number of states with definite values of $\mathbf{L}^2$ and L_z agrees with the number of states you found in part (i). [4 marks]

iii. Without doing any explicit calculations, explain how the operator $\widehat{L}_{\rm up}$ in part (b) could be used to construct all states with the largest value of L.

[4 marks]

iv. The atom is placed in a weak magnetic field B. Describe quantitatively what happens to the level with the largest value of L. [3 marks]

4. An electron with zero orbital angular momentum is placed in a magnetic field that points in the $+z$ direction. The magnetic moment of the electron is

$$\widehat{\boldsymbol{\mu}} = -\frac{e}{m_e}\widehat{\mathbf{S}}.$$

At time $t = 0$, the electron is in the state

$$\chi(0) = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,\alpha_z + \beta_z\right).$$

(a) Show that the spinor

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of the spin operator $\widehat{S}_x$ with eigenvalue $S_x = +\hbar/2$. Find the probabilty that a measurement of the spin of the electron in the state $\chi(0)$ gives $S_x = +\hbar/2$. [8 marks]

(b) Write down the time-dependent Schrödinger equation describing the interaction of the electron's spin with the magnetic field. Hence show that, at time t, the spin state of the electron is

$$\chi(t) = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\omega t}\alpha_z + \mathrm{e}^{+\mathrm{i}\omega t}\beta_z\right),$$

and find an expression for ω. [10 marks]

(c) Find the expectation values of $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ as functions of time for this state. Hence describe qualitatively the motion of the spin of this electron. [7 marks]

NUMERICAL AND BOTTOM-LINE ANSWERS

1. (a)
$$E_{n_1\,n_2\,n_3} = \frac{\hbar^2\pi^2}{2\,m\,a^2}\left(n_1^2 + n_2^2 + \frac{n_3^2}{3}\right), \qquad n_i = 1,\,2,\,3,\,\ldots$$
   (b) No numerical answer
   (c) $\Delta E = 3.9 \times 10^{-3}$ eV
   (d)
$$E_{\text{so}} = \frac{A\,\hbar^2}{2}\big[j(j+1) - l(l+1) - s(s+1)\big]$$
   (e) No numerical answer

2. (a) No numerical answer
   (b) No numerical answer
   (c) Edges of classically forbidden zone are at $x = \pm b$ where $E = V(b) = V_0 - k\,b^2/2$
   Tunnelling factor is
$$T = \exp\left[-2\int_{-b}^{b}\sqrt{\frac{2\,m}{\hbar^2}\,\frac{k}{2}\big(b^2 - x^2\big)}\;\mathrm{d}x\right]$$
   which can be put into given form by changing variable to $y = x/b$
   (d) Parabolic barrier: $T \simeq 6 \times 10^{-11}$
   Square barrier: $T \simeq 9 \times 10^{-14}$ (higher barrier on average)

3. (a) No numerical answer
   (b) No numerical answer
   (c) i. $5 \times 7 = 35$ states
   ii. $L = 1,\,2,\,3,\,4,\,5$
   so $3 + 5 + 7 + 9 + 11 = 35$ states
   iii. No numerical answer
   iv. Splits into 11 equally spaced levels with splitting
$$\Delta E = \frac{e\,\hbar}{2\,m}\,B$$

4. (a)

$$P = \left|\alpha_x^\dagger \chi(0)\right|^2 = \frac{1}{6}\big(3 + 2\sqrt{2}\big)$$

(b)

$$\mathrm{i}\,\hbar\,\frac{\partial \chi}{\partial t} = \widehat{H}\,\chi$$

where

$$\widehat{H} = \frac{e\,\hbar\,B}{2\,m}\,\sigma_3$$

Find the (two) separable solutions, form a linear superposition of them, and use the initial condition to fix the coefficients

Frequency

$$\omega = \frac{e\,B}{2\,m}$$

(c)

$$\begin{aligned}
\langle S_x \rangle &= \chi(t)^\dagger \widehat{S}_x \chi(t) \\
&= \hbar\,\frac{\sqrt{2}}{3}\cos(2\omega t) \\
\langle S_y \rangle &= \hbar\,\frac{\sqrt{2}}{3}\sin(2\omega t) \\
\langle S_z \rangle &= \frac{\hbar}{6}
\end{aligned}$$