In this paper α_z and β_z denote the eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the value for the Bohr magneton, $\mu_B = e\hbar/2m_e = 5.8 \times 10^{-5}$ eV/T.

1. (a) An electron is trapped inside a sphere of radius R by a potential that is zero for $r < R$ and infinite for $r > R$. Write down its ground state energy, and state the boundary conditions that lead to this. Calculate this energy for a sphere of radius $R = 3$ nm, made of solid CdTe in which electrons have an effective mass $m^* = 0.096\, m_e$. [5 marks]

   (b) Without any detailed algebra, explain the form of the expression

   $$T \simeq \exp\left[-2\int_a^b \sqrt{\frac{2m}{\hbar^2}\Big(V(x) - E\Big)}\,\mathrm{d}x\right]$$

   for the probability of a particle of energy E to tunnel through a smoothly varying barrier $V(x)$. Your answer should define the points a, b. [5 marks]

   (c) Write down the spin operators $\widehat{S}_x$, $\widehat{S}_y$, and $\widehat{S}_z$ in terms of the Pauli matrices and hence verify that

   $$\left[\widehat{S}_z, \widehat{S}_y\right] = -\mathrm{i}\hbar\widehat{S}_x.$$

   [5 marks]

   (d) A W boson has an intrinsic spin $S = 1$ and is in a state with orbital angular momentum $L = 1$. What are its possible values of the quantum numbers J and M_J describing its total angular momentum? [5 marks]

   (e) The spins of two electrons are in the state

   $$\phi(1,2) = \frac{1}{2}\Big(\alpha_x(1)\,\alpha_x(2) - \alpha_x(1)\,\beta_x(2) - \beta_x(1)\,\alpha_x(2) + \beta_x(1)\,\beta_x(2)\Big).$$

   Is this an "entangled" state? Justify your answer. [5 marks]

2. (a) Define the terms “quantum dot”, “quantum wire” and “quantum well”.

[5 marks]

(b) A quantum wire is constructed with a rectangular cross section, with sides a and $2a$ in the x and y directions. The wire is surrounded on all sides by an insulator which you can treat as providing an infinite potential energy for electrons. Consider a long piece of this wire, with length $L \gg a$ in the z direction. Write down the the general form of the energy eigenvalues of an electron in this wire. What are the energy levels of the ground state and the first two excitations of the xy motion? [7 marks]

(c) Find the density of states for electrons in the wire, $\mathrm{d}n/\mathrm{d}k_z$, as a function of k_z, the wavenumber in the z direction. Convert this to the density of states as a function of energy, $\mathrm{d}n/\mathrm{d}E$, for a particular eigenvalue of the xy motion. Sketch the total density of states as a function of energy. [13 marks]

3. (a) Explain briefly the origin of the spin-orbit interaction,

$$\widehat{H}_{so} = A\,\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}},$$

in a one-electron atom. [5 marks]

(b) Show that eigenstates of the spin-orbit interaction $\widehat{H}_{so}$ do not in general have definite values for the quantum numbers m_l and m_s. Show also that these eigenstates can have definite values for m_j. You may assume the commutation rules for the components of any angular momentum operator, e.g.

$$\left[\widehat{L}_x, \widehat{L}_y\right] = \mathrm{i}\hbar\,\widehat{L}_z.$$

Rewrite the spin-orbit interaction in a form that shows that eigenstates of $\widehat{H}_{so}$ can have a definite value for the quantum number j. [10 marks]

(c) The outer electron of a neutral potassium atom is in a $5g$ orbital. Show that there are two possible values for its energy, and find the difference between these values in terms of the constant A. [5 marks]

(d) Describe what happens to these two $3g$ levels when the atom is placed in a weak magnetic field B. Calculate the splitting between the states of the lower level for a field of strength $B = 1$ T. [5 marks]

4. (a) A quantum device is constructed so that it can copy the circular polarisation state of a photon by emitting a second photon with the same circular polarisation. If R and L denote the right- and left-handed polarisation states, the action of the copier can be represented by

$$\widehat{X}\, R(1) = R(1)\, R(2), \qquad \text{and} \qquad \widehat{X}\, L(1) = L(1)\, L(2).$$

Show that the device fails to produce an identical copy of a photon that is in a general polarisation state. Show also that the resulting pair of photons is in an entangled state. [7 marks]

(b) Alice and Bob prepare pairs of photons in the entangled state

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\Big(R(1)R(2) + L(1)L(2)\Big),$$

which they use to encrypt messages by sending photon 1 to Alice and photon 2 to Bob. Each of them measures their photons with analysers that distinguish vertical and horizontal polarisations, with eigenstates

$$V = \frac{1}{\sqrt{2}}\Big(R + L\Big), \qquad \text{and} \qquad H = \frac{1}{\sqrt{2}}\Big(R - L\Big).$$

Show that Bob is able to deduce the results of Alice's measurements.

[5 marks]

(c) Eve breaks into their communications and uses the device in part (a) to copy the polarisation of Bob's photons, leaving them with three photons in the state,

$$\widehat{X}\Psi(1,2) = \frac{1}{\sqrt{2}}\Big(R(1)R(2)R(3) + L(1)L(2)L(3)\Big),$$

where photon 3 is Eve's copy. Suppose that all three people measure their photons with analysers that distinguish vertical and horizontal polarisations, as described above. Show that Eve gets no information about the state of polarisation of Bob's photons. [8 marks]

(d) Explain why Alice and Bob will suspect that they have an eavesdropper.

[5 marks]

NUMERICAL AND BOTTOM-LINE ANSWERS

1. (a) 0.46 eV

   (b) No numerical answer

   (c) No numerical answer

   (d) $J = 0,\ 1$ or 2; $M_J = -2,\ -1,\ 0,\ +1,\ +2$

   (e) Not entangled

2. (a) No numerical answer

   (b) General eigenvalue:

$$E_{n_1 n_2 n_3} = \frac{\hbar^2\,\pi^2}{2\,m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{4a^2} + \frac{n_3^2}{L^2}\right)$$

   Lowest eigenvalues:

$$\begin{aligned} E_{111} &= \frac{\hbar^2\,\pi^2}{2\,m}\left(\frac{5}{4a^2} + \frac{1}{L^2}\right) \\ E_{121} &= \frac{\hbar^2\,\pi^2}{2\,m}\left(\frac{2}{a^2} + \frac{1}{L^2}\right) \\ E_{131} &= \frac{\hbar^2\,\pi^2}{2\,m}\left(\frac{13}{4a^2} + \frac{1}{L^2}\right) \end{aligned}$$

   (c) Density of states (for a single transverse mode) in terms of k_z:

$$\frac{\mathrm{d}n}{\mathrm{d}k_z} = \frac{2L}{\pi}$$

   Density of states in terms of E:

$$\frac{\mathrm{d}n}{\mathrm{d}E} = \frac{L}{\pi}\sqrt{\frac{2\,m}{\hbar^2(E - E_{n_1 n_2})}}$$

   Your plot should show a set of spikes where, at each $E = E_{n_1 n_2}$, the density of states jumps to infinity and then falls off as $1/\sqrt{E - E_{n_1 n_2}}$.

3. (a) No numerical answer

   (b) Your answer should show clearly that $\left[\widehat{H}_{so}, \widehat{L}_z\right] \neq 0$ (and similarly for $\widehat{S}_z$).
   $\widehat{H}_{so} = \frac{A}{2}\left(\widehat{\mathbf{J}}^2 - \widehat{\mathbf{L}}^2 - \widehat{\mathbf{S}}^2\right)$

   (c) $j = 7/2$ or $9/2$
   Splitting $\Delta E_{so} = \frac{9}{2}\,A\,\hbar^2$

   (d) $\Delta E_{mag} = 5.2 \times 10^{-5}$ eV

4. (a) No numerical answer

   (b) In terms of V and H states:

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\Big(V(1)V(2) + H(1)H(2)\Big)$$

   (c) Similarly:

$$\begin{aligned}\widehat{X}\Psi(1,2) \quad = \quad & \frac{1}{2}\Big(V(1)V(2)V(3) + V(1)H(2)H(3) \\ & +H(1)V(2)H(3) + H(1)H(2)V(3)\Big)\end{aligned}$$

   (d) No numerical answer