Here α_z and β_z denote the eigenstates of $\widehat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the fact that $ec = 0.3$ eV/nm/T.

1. (a) Estimate the probability for a neutron to tunnel through a square barrier 2 fm wide when the energy of the neutron is 40 MeV below the top of the barrier. [5 marks]

   (b) An electron is in the ground state of a one-dimensional harmonic oscillator well. It is subject to a weak electric field which leads to an additional term

   $$\widehat{H}_1 = e\mathcal{E}\,x$$

   in the electron's Hamiltonian. Show that to first order in $\mathcal{E}$, the electric field has no effect on the energy of the electron. [The ground-state wave function of the harmonic oscillator has the form $\psi_0(x) = N\exp(-x^2/(2b^2))$ where N and b are constants.] [5 marks]

   (c) Show that

   $$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$$

   is an eigenstate of the spin operator $\widehat{S}_y$ and find its eigenvalue.
   Show also that α_y is not an eigenstate of $\widehat{S}_x$. [5 marks]

   (d) The two electrons in an atom are excited into a state where one has angular momentum quantum number $j_1 = 3/2$ and the other has $j_2 = 9/2$. What are the possible values of J (the total angular momentum quantum number)? [5 marks]

   (e) The spins of two electrons are in the state

   $$\psi(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\,\beta_z(2) - \beta_z(1)\,\alpha_z(2)\Big).$$

   Why do we describe this as an "entangled" state? [5 marks]

2. A quantum dot consists of a cuboid block of undoped silicon, with sides of length a, a and $a/\sqrt{2}$. It is surrounded by insulating walls, where the potential energy of the electron can be assumed to be infinite. The potential energy of an electron inside the dot is zero.

   (a) Find the first three energy levels of a single electron in the dot. What are the degeneracies of these levels? What "magic" numbers of electrons correspond to the first three closed shells? [10 marks]

   (b) Find the ground-state energy of the electron for a block with $a = 10$ nm, taking the effective mass of an electron in silicon to be $m^* = 0.2\,m_e$. Where in the electromagnetic spectrum would you expect to see radiation from transitions between states in this dot? [4 marks]

   (c) The dot is placed between two electrical contacts (the drain and source) and is separated from them by thin insulating layers. It is also surrounded by a gate, which can be used to change the constant potential inside the dot. The drain and source are held at the same voltage. Explain why the dot will conduct current only for certain discrete values of the gate voltage. Describe the pattern you expect for the spacings between these values. [11 marks]

3. An electron with zero orbital angular momentum is placed in a magnetic field that points in the $+z$ direction. The magnetic moment of the electron is

$$\widehat{\boldsymbol{\mu}} = -\frac{e}{m_e}\,\widehat{\mathbf{S}}.$$

   At time $t = 0$, the electron is in the eigenstate of $\widehat{S}_y$,

$$\alpha_y = \frac{1}{\sqrt{2}}\left(\alpha_z + \mathrm{i}\,\beta_z\right).$$

   (a) Write down the time-dependent Schrödinger equation describing the interaction of this moment with the field. Hence show that, at time t, the spin state of the electron is

$$\chi(t) = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\omega t}\alpha_z + \mathrm{i}\,\mathrm{e}^{+\mathrm{i}\omega t}\beta_z\right),$$

   and find an expression for ω. [10 marks]

   (b) Show that the expectation value of $\widehat{S}_y$ in this state at time t is given by

$$\langle \widehat{S}_y \rangle = \frac{\hbar}{2}\cos(2\omega t).$$

   [8 marks]

   (c) Find the probability for a measurement of $\widehat{S}_y$ at time t to give the value $+\hbar/2$. [7 marks]

4. (a) An atom is in an energy level with total orbital angular momentum quantum number L and total spin quantum number S. These are coupled to a total angular momentum quantum number J. The atom is placed in a weak magnetic field.

Write down an expression for the magnetic energies of the states corresponding to this level. Without giving a detailed algebraic derivation, explain the origin of the Landé g-factor in this expression. Why does the magnetic field need to be weak for this to be a good approximation? [8 marks]

(b) The low-energy states of Ti^{3+} ions have $L = 2$ and $S = 1/2$. The spin-orbit interaction Hamiltonian for the ions can be taken to have the form

$$\widehat{H}_{\mathrm{so}} = A\,\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}},$$

where A is a constant. List the possible values of J and determine the resulting energy shifts for each of these states. [7 marks]

(c) Sketch the energy-level diagram of this system in a weak magnetic field. Indicate the quantum numbers of all levels. Find the splittings between the levels for a magnetic field of 2 T. [6 marks]

(d) Ti^{3+} ions are placed in a magnetic field that is so strong that the effect of the spin-orbit interaction can be neglected. Sketch the energy-level diagram for the states with $L = 2$ and $S = 1/2$ in this field.

[4 marks]

NUMERICAL AND BOTTOM-LINE ANSWERS

1. (a) $T \simeq 4 \times 10^{-3}$

   (b) No numerical answer

   (c) Eigenvalue $S_y = +\hbar/2$

   (d) $J = 3,\ 4,\ 5$ or 6

   (e) No numerical answer

2. (a) Lowest eigenvalues:

$$E_{111} = 4\,\frac{\hbar^2\,\pi^2}{2\,m\,a^2}, \qquad E_{211} = E_{121} = 7\,\frac{\hbar^2\,\pi^2}{2\,m\,a^2}, \qquad E_{221} = E_{112} = 10\,\frac{\hbar^2\,\pi^2}{2\,m\,a^2}$$

   Orbital degeneracies: 1, 2, 2
   Magic numbers: 2, 6, 10

   (b) $E_{111} \simeq 8 \times 10^{-2}$ eV
   Infrared radiation

   (c) No numerical answer

3. (a) $\omega = \dfrac{e\,B}{2\,m}$

   (b) No numerical answer

   (c) $P(t) = \cos^2(\omega\,t)$

4. (a) No numerical answer

   (b) $J = \dfrac{3}{2}$ or $\dfrac{5}{2}$

$$E_{3/2} = -\,\frac{3}{2}\,A\hbar^2, \qquad E_{5/2} = +A\hbar^2$$

   (c) Your diagram should show the $J = 5/2$ level splitting into six equally spaced levels ($M_J = +5/2, \ldots, -5/2$) when the field is switched on. The $J = 3/2$ level should split into four levels. All the $J = 5/2$ levels should lie above all the $J = 3/2$ ones.

$$\Delta E_{3/2} = \frac{4}{5}\,\frac{e\,\hbar}{2\,m}\,B \simeq 10^{-4}\ \text{eV}$$

$$\Delta E_{5/2} = \frac{6}{5}\,\frac{e\,\hbar}{2\,m}\,B \simeq 1.4 \times 10^{-4}\ \text{eV}$$

   (d) Your diagram should show levels with definite values for M_L and M_S (not J and M_J). Assuming $g = 2$, there should be seven equally spaced levels, the highest with $M_L = +2$ and $M_S = +1/2$, the lowest with $M_L = -2$ and $M_S = -1/2$. The middle three should be doubly degenerate.