You may use the standard set of Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You may also use the Landé g-factor,

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)},$$

and the fact that $ec = 0.3$ eV/nm/T.

1. (a) Estimate the probability for an electron to tunnel through a square barrier 1 nm wide when the energy of the electron is 2 eV below the top of the barrier. [5 marks]

   (b) An electron is trapped inside an ellipsoidal quantum dot by the potential

   $$V(x, y, z) = \frac{1}{2} k \left(x^2 + y^2 + 4z^2\right).$$

   Write down its first three energy levels and state their degeneracies. [5 marks]

   (c) Write down the spin operators $\widehat{S}_x$, $\widehat{S}_y$, and $\widehat{S}_z$ in terms of the Pauli matrices and hence verify that

   $$\left[\widehat{S}_z, \widehat{S}_x\right] = \mathrm{i}\hbar \widehat{S}_y.$$

   [5 marks]

   (d) The quarks in an excited baryon have a total orbital and spin angular-momentum quantum numbers $L = 1$ and $S = 3/2$, respectively. What are the allowed values of J (the quantum number that gives the eigenvalue the total angular momentum)? [5 marks]

   (e) Find the interaction energy (in eV) of the spin of an electron with the magnetic field of the world's strongest magnet, $B = 45$ T. [5 marks]

2. (a) Without any detailed algebra, explain the origin of the formula

$$T \simeq \exp\left[-2\int_a^b \sqrt{\frac{2m}{\hbar^2}\Big(V(x)-E\Big)}\,\mathrm{d}x\right]$$

for the probability of a particle of energy E to tunnel through a barrier $V(x)$. Your answer should state briefly the conditions under which this is a good approximation, and define the points a, b. [8 marks]

(b) In nuclear α decay, a ^{4}He nucleus (atomic number $Z_1 = 2$) tunnels out through the repulsive Coulomb potential between it and the final nucleus (atomic number Z_2). Assuming that the attractive nuclear force forms a square well of radius R_N, find an expression for the tunnelling probability for an alpha particle of energy E to escape from the nucleus. [9 marks]

[You may use the standard integral,

$$\int \left(\frac{1}{x}-1\right)^{1/2} \mathrm{d}x = \sqrt{x(1-x)} - \cos^{-1}\sqrt{x} + C.\ ]$$

(c) Bismuth isotopes can undergo α decay to thallium ($Z_2 = 81$). One isotope, bismuth-211, emits α particles with energy 6.8 MeV and has a lifetime of 150 s. Another, bismuth-209, emits α particles with energy 3.2 MeV, and has a lifetime of 6.0×10^{26} s. You may assume that the radii of the nuclei involved are small compared to any other relevant distances in the problem, and you should ignore the recoil of the heavy nuclei. Use your result above to explain this factor $\sim 10^{24}$ difference in lifetimes. [8 marks]

3. (a) Explain the origin of the spin-orbit coupling in a one-electron atom, commenting on the sign of the interaction. [6 marks]

(b) The spin-orbit interaction energy operator has the form

$$\widehat{H}_{so} = f(r)\widehat{\mathbf{L}} \cdot \widehat{\mathbf{S}}.$$

Rewrite this in a form that shows that eigenstates of the total angular momentum operator $\widehat{\mathbf{J}}^2$ are not mixed by this interaction. [3 marks]

(c) A potassium atom has one electron outside a closed shell. Its lowest $l = 1$ level is split into two levels with an energy difference of 7.2×10^{-3} eV. State the values of the quantum number j for these levels, indicating which has the lower energy. Determine the value (in eV) of the radial matrix element

$$\mathcal{E}_{so} = \langle f(r) \rangle \hbar^2.$$

[8 marks]

(d) Sketch the energy-level diagram of this $l = 1$ system in a weak magnetic field. Indicate the quantum numbers of all levels. Find the splittings between the levels for a magnetic field of 0.5 T. [8 marks]

4. (a) The normalised eigenvectors of the operator $\widehat{S}_y$ for a spin-1/2 particle are

$$\alpha_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \quad \text{and} \quad \beta_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}.$$

Verify that α_y is indeed an eigenvector and that it is normalised. [4 marks]

(b) Two electrons are in a state $\psi(1,2)$ with a total spin of zero. Write down an expression for this state in terms of $\alpha_z(i)$ and $\beta_z(i)$, the eigenvectors of $\widehat{S}_z$ for the two electrons ($i = 1, 2$). Express the state $\psi(1,2)$ in terms of the vectors $\alpha_y(i)$ and $\beta_y(i)$ given above. Explain briefly why we say that these two electrons are in an "entangled" state. [13 marks]

(c) Explain how Alice and Bob can use pairs of electrons prepared in the state $\psi(1,2)$ to set up a secure key for the exchange of encrypted messages. How could they detect whether a third party had been eavesdropping? [8 marks]

NUMERICAL AND BOTTOM-LINE ANSWERS

1. (a) $T \simeq 7 \times 10^{-7}$ (or 5×10^{-7} if you're more accurate than I am with the numbers)

   (b)

$$E_{000} = 2\hbar\omega \qquad d = 1$$
$$E_{100} = 3\hbar\omega \qquad d = 2$$
$$E_{200} = 4\hbar\omega \qquad d = 4$$

   (c) No numerical answer

   (d) $J = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}$

   (e) $|E| = 2.7 \times 10^{-3}$ eV

2. (a) No numerical answer

   (b)

$$T \simeq \exp\left[-2\,\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{2m}{E}}\left(\cos^{-1}\sqrt{\frac{r_N}{r_c}} - \sqrt{\frac{r_N}{r_c}\left(1 - \frac{r_N}{r_c}\right)}\right)\right],$$

   where $r_c = Z_1 Z_2 e^2/4\pi\epsilon_0 E$

   (c) If $r_N \ll r_c$, $\cos^{-1}\sqrt{r_N/r_c} \simeq \pi/2$ and

$$T \simeq \exp\left[-\pi\,\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{2m}{E}}\right].$$

   For $Z_1 = 2$, $Z_2 = 81$, $m \simeq 4m_N$ and $E = 6.8$ MeV, $T_{211} \simeq 2 \times 10^{-54}$.
   Similarly for $E = 3.2$ MeV, $T_{209} \simeq 6 \times 10^{-79}$.
   The ratio of the decay rates is thus $T_{211}/T_{209} \simeq 3 \times 10^{24}$.

3. (a) No numerical answer

   (b) Use $\widehat{\mathbf{J}}^2 = \widehat{\mathbf{L}}^2 + \widehat{\mathbf{S}}^2 + 2\,\widehat{\mathbf{L}}\cdot\widehat{\mathbf{S}}$.

   (c) $j = \frac{1}{2},\ \frac{3}{2}$, with $j = \frac{1}{2}$ being the lower level.
   $\mathcal{E}_{\text{so}} = 4.8 \times 10^{-3}$ eV

   (d) $j = \frac{3}{2}$ splits into 4 equally spaced levels, $m_j = +\frac{3}{2}, \cdots, -\frac{3}{2}$.
   $j = \frac{1}{2}$ splits into 2 levels, $m_j = \pm\frac{1}{2}$.
   Within each set the levels are ordered by their m_j values.
   The levels are split by

$$\begin{aligned}\Delta E &= \frac{e g_j \hbar}{2m}\,B \\ &\simeq \begin{cases} 2 \times 10^{-5}\ \text{eV} & \text{for } j = \frac{1}{2} \\ 4 \times 10^{-5}\ \text{eV} & \text{for } j = \frac{3}{2} \end{cases}.\end{aligned}$$

4. (a) No numerical answer

   (b)

$$\psi(1,2) = \frac{1}{\sqrt{2}}\Big(\alpha_z(1)\,\beta_z(2) - \beta_z(1)\,\alpha_z(2)\Big)$$

   Using

$$\alpha_z = \frac{1}{\sqrt{2}}\Big(\alpha_y + \beta_y\Big), \qquad \alpha_z = -\frac{\mathrm{i}}{\sqrt{2}}\Big(\alpha_y - \beta_y\Big),$$

   gives

$$\psi(1,2) = \frac{\mathrm{i}}{\sqrt{2}}\Big(\alpha_y(1)\,\beta_y(2) - \beta_y(1)\,\alpha_y(2)\Big)$$

   (c) No numerical answer