PHYS 20171 MATHEMATICS OF WAVES AND FIELDS

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The aim of this course is to develop some of techniques needed to solve linear partial differential equations (PDE's). These equations appear in many areas of physics and describe waves and fields which can vary in one or more space dimensions and in time. They include:

- Laplace's equation

  $$\nabla^2\phi(\mathbf{r}) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

  The field $\phi(\mathbf{r})$ could be, for example, the electrostatic potential in a region of space without electric charge, or the steady-state distribution of temperature inside some body.

- The ordinary (or nondispersive) wave equation

  $$\nabla^2\phi(\mathbf{r}) = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

  This describes waves with a constant speed c. These could be sound waves, if $\phi(\mathbf{r}, t)$ is the displacement of a vibrating string or membrane or medium. They could also be electromagnetic waves, such as light or radio, if $\phi(\mathbf{r}, t)$ is one of the components of the electric field.

- The Schrödinger equation

  $$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\mathbf{r})\psi(\mathbf{r}, t) = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}$$

  This is the central equation of quantum mechanics. It describes the quantum mechanical wave for a particle of mass m moving in a potential $V(\mathbf{r})$. The wave function $\psi(\mathbf{r}, t)$ gives the probability amplitude for finding the particle at the point $\mathbf{r}$ at time t.

- The heat-flow (or diffusion) equation

  $$\nabla^2\phi(\mathbf{r}, t) = \frac{1}{D}\frac{\partial\phi}{\partial t}$$

  This describes the flow of heat inside a body with no internal source of heat; the field $\phi(\mathbf{r}, t)$ is the (time-dependent) temperature distribution inside the body.

There are three important mathematical tools which we shall be using to tackle this kind of equation. The first of these is separation of variables. Provided the boundary conditions on our PDE are simple enough, we can find special solutions that can be written as products of functions of one variable, for example: $\phi(x, y, z) = X(x)Y(y)Z(z)$.

The separated functions, like $X(x)$, $Y(y)$ and $Z(z)$, satisfy ordinary differential equations (ODE's) which contain unknown constants. These can be solved only for special values of the constants, known as eigenvalues. The corresponding solutions are called eigenfunctions. Physically these correspond to the normal modes of an oscillating system.

Finally we use these eigenfunctions as basis functions to build a general solution to the PDE as a linear superposition of the separable ones. The resulting series contains an infinite number of constants which need to be determined from the initial conditions.

The amount of each basis function needed can be found from a straightforward integral, provided the set of basis functions has a property known as orthogonality. An example is the Fourier series, where we add sines and cosines to form a general wave. For waves that can spead out to infinity the series becomes an integral, known as a Fourier transform.

We shall also meet other important eigenvalue problems (ODE's with boundary conditions) in the context of waves and fields in two or three dimensions. All of these lead to sets of orthogonal basis functions. They include: Bessel functions, spherical Bessel functions and Legendre polynomials. You will meet these again (and others like them) in quantum mechanics and in studying electromagnetic waves.

Having separated variables and found the set of eigenfunctions for our problem, we build a general solution out of the separable ones. The coefficient of each separable solution can then be determined from the initial conditions on our physical problem.

Recommended book

The recommended book for this course is:

- M. L. Boas, *Mathematical methods in the physical sciences*, 3rd edn., (Wiley, 2006).

The relevant material is contained in: Chapter 7, Sections 5, 10 and 11 of Chapter 8, Chapter 12, and Chapter 13. You are strongly advised to get hold of a copy of this book since it will provide the necessary mathematical backup for your physics courses over the next two or three years (except for the most abstruse theoretical options). You may find quite a few second-hand copies of the older (2nd) edition of this book. It covers all the same material, but note that the material on Fourier transforms is organised differently. The course outline below indicates the equivalent sections of the 2nd edition, where these differ.

A good alternative (which some people prefer) is: K. F. Riley, M. P. Hobson and S. J. Bence, *Mathematical Methods for Physics and Engineering* (Cambridge, 1997), Chapters 10, 11, 13.1, 14, 15, 16 and 17. Take a look at it if you don't get on with Boas. Another useful book, on applications of these ideas to PDE's, is: G. Stephenson, *Partial differential equations for scientists and engineers* (Imperial College, 1996).

Examples sheets

I know you've heard this before, but the only way to become familiar with mathematical techniques is to *use* them. The questions on the examples sheets for tutorials will give you some practice. You should also make use of the large sets of practice problems at the end of each section of Boas's book. (The book also contains lots of helpful worked examples.)

Webpage

I have collected links to a number of relevant sites on a webpage for this course:

`http://theory.ph.man.ac.uk/∼mikeb/lecture/pc217/index.html`

Please let me know of any other sites which you find useful.

Course outline

References to book chapters or sections, as follows:

B3 Boas, 3rd edition

B2 Boas, 2nd edition

R+ Riley, Hobson and Bence

Examples of partial differential equations in physics

B3 and **B2** 13.1; **R+** 16.1

0. **Ordinary differential equations** (∼ 1 lecture)

   B3 and **B2** 8.5, 2.9, 2.11, 2.12; **R+** 13.1

   First-order, linear

   Second-order, linear

   Complex exponentials

1. **Wave problems in one dimension** (∼ 2 lectures)

   B3 and **B2** 13.2 (separation of variables), 13.4; **R+** 16.1.1, 17.1, 17.2

   Separation of variables

   Normal modes of a string: eigenfunctions and eigenvalues

   General motion of a string

2. **Fourier series** (∼ 4 lectures)

   B3 and **B2** 7.1–7.11; **R+** 10

   Orthogonality and completeness of sines and cosines

   Fourier coefficients

   Complex exponential form of Fourier series

   Initial conditions on PDE's

3. **Other PDE's** ($\sim$ 2 lectures)

   B3 and **B2** 13.2, 13.3; **R+** 16.1.2, 16.1.3, 17.2

   Laplace's equation

   The heat-flow equation

4. **Integral transforms** ($\sim$ 3 lectures)

   B3 7.12, 8.10, 8.11; **B2** 15.4, 15.5, 15.7; **R+** 11.1

   Fourier transform

   Convolutions

   Wave packets and dispersion

5. **Series solution of ODE's** ($\sim$ 4 lectures)

   B3 and **B2** 1.6C, 1.10, 1.12, 12.2, 12.6, 12.7–12.9, 12.11, 12.12;
   R+ 3.3, 3.6, 14.2, 14.3, 14.6, 14.7

   Taylor series

   Legendre polynomials and related functions

   Bessel functions

   Orthogonal sets of eigenfunctions

   Legendre series

6. **Problems in two and three dimensions** ($\sim$ 6 lectures)

   B3 and **B2** 13.5–13.7; **R+** 17.3

   Normal modes of a square membrane; degeneracy

   Wave guide

   Normal modes of a circular and spherical systems

   Heat flow and Laplace's equation in circular and spherical systems

Mike Birse (September 2007)