A Bestiary of Special Functions

This provides a brief summary of some of the special functions that you are likely to meet in your physics courses. More details of their properties can be found in Chapter 12 of Boas. All of them satisfy equations that can be written in the form of eigenvalue problems:

$$\widehat{L}y(x) = \lambda y(x),$$

where $\widehat{L}$ is a linear differential operator.

1. **Plane waves:** $\cos(kx)$, $\sin(kx)$, $\exp(ikx)$

   Equation: $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + k^2y = 0$$

   Eigenvalue: $\lambda = k^2$

   Range: depends on the boundary conditions imposed

   Applications: numerous one-dimensional problems, such as:
   waves on a string, quantum particle in a box;
   also dependence on the azimuthal angle (ϕ) in polar coordinates

2. **Legendre polynomials:** $P_l(x)$

   Equation: $$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0$$

   Eigenvalue: $\lambda = l(l+1)$

   Range: $-1 \leq x \leq +1$ (regular at $x = \pm 1$)

   Examples: $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1)$

   Applications: with the change of variable $x = \cos\theta$, the dependence on polar angle θ in spherical polar coordinates, where the system is independent of ϕ ($m = 0$)

3. **Associated Legendre functions:** $P_l^m(x)$

Equation: $$(1-x^2)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{m^2}{1-x^2}\,y + l(l+1)y = 0$$

Eigenvalue: $\lambda = l(l+1)$

Range: $-1 \le x \le +1$ (regular at $x = \pm 1$)

Examples: $P_0^0(x) = 1,\ P_1^0(x) = x,\ P_1^1 = \sqrt{1-x^2}$

Applications: with the change of variable $x = \cos\theta$, the dependence on polar angle θ in spherical polar coordinates; these generalise the Legendre polynomials to cases with $m \neq 0$;
spherical harmonics are defined, up to a normalisation constant, by $Y_{lm}(\theta,\phi) \propto P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$

4. **Hermite polynomials:** $H_n(x)$

Equation: $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x^2y + (2n+1)y = 0$$

Eigenvalue: $\lambda = 2n+1$

Range: $-\infty < x < +\infty$

Examples: $H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2$

Applications: wave functions of the quantum harmonic oscillator

5. **Bessel functions:** $J_m(kx)$ (regular), $N_m(kx)$ (irregular, also written $Y_m(kx)$)

Equation: $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{m^2}{x^2}\,y + k^2y = 0$$

Eigenvalue: $\lambda = k^2$

Range: $0 \leq x < +\infty$ for $J_m(kx)$ (regular at $x = 0$)
$0 < x < +\infty$ for $N_m(kx)$ (singular at $x = 0$)

Examples: $J_0(kx) = 1 - \frac{1}{4}(kx)^2 + \cdots,\quad J_2(kx) = \frac{1}{2^2\,2!}(kx)^2 - \frac{1}{2^3\,3!}(kx)^4 + \cdots$

Applications: radial dependence of free waves in plane or cylindrical polar coordinates; diffraction from a circular aperture

6. **Spherical Bessel functions:** $j_l(kx)$ (regular), $y_l(kx)$ (irregular)

Equation: $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{l(l+1)}{x^2}\,y + k^2y = 0$$

Eigenvalue: $\lambda = k^2$

Range: $0 \leq x < +\infty$ for $j_l(kx)$ (regular at $x = 0$)
$0 < x < +\infty$ for $y_l(kx)$ (singular at $x = 0$)

Examples: spherical Bessel functions can all be written in terms of cos, sin and inverse powers of x:

$$j_0(kx) = \frac{\sin(kx)}{kx} \text{ (a.k.a. sinc}(kx)), \quad y_0(kx) = -\frac{\cos(kx)}{kx}$$

Applications: radial dependence of free waves in spherical polar coordinates

7. **Laguerre polynomials:** $L_n(x)$

Equation: $$x\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + ny = 0$$

Eigenvalue: $\lambda = n$

Range: $0 \leq x < +\infty$

Applications: used to define the associated Laguerre polynomials

$$L_n^k(x) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}\, L_n(x);$$

numerical evaluation of integrals

8. **Associated Laguerre polynomials:** $L_n^k(x)$

Equation: $$x\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (k+1-x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + ny = 0$$

Eigenvalue: $\lambda = n$

Range: $0 \leq x < +\infty$

Applications: radial wave functions of the Hydrogen atom

$$R_{nl}(r) \propto r^l\, \mathrm{e}^{-r/na_0} L_{n+l}^{2l+1}(2r/na_0)$$

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