Problems and solutions for Fourier transforms and δ-functions

1. Prove the following results for Fourier transforms, where F.T. represents the Fourier transform, and F.T. $[f(x)] = F(k)$:
a) If $f(x)$ is symmetric (or antisymmetric), so is $F(k)$: i.e. if $f(x) = \pm f(-x)$ then $F(k) = \pm F(-k)$.
b) If $f(x)$ is real, $F^*(k) = F(-k)$.
c) If $f(x)$ is real and symmetric (antisymmetric), $F(k)$ is real and symmetric (imaginary and antisymmetric).
d) F.T. $[f(\kappa x)] = \dfrac{1}{|\kappa|} F\left(\dfrac{k}{\kappa}\right)$.
e) F.T. $[f(x+a)] = \mathrm{e}^{ika}F(k)$.
f) F.T. $[\mathrm{e}^{\alpha x} f(x)] = F(k + i\alpha)$ (for real or complex α.)
g) F.T. $[F(x)] = f(-k)$.
h) F.T. $[\delta(x - x_0)] = \mathrm{e}^{-ikx_0}/\sqrt{2\pi}$
i) F.T. $[\mathrm{e}^{ik_0x}] = \sqrt{2\pi}\delta(k - k_0)$

a)

$$F(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i(-k)x}\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x')\mathrm{e}^{-ikx'}\mathrm{d}x'$$

where $x' = -x$. So if $f(-x) = \pm f(x)$, then $F(k) = \pm F(-k)$.

b)

$$F(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{ikx}\mathrm{d}x = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(x)\mathrm{e}^{-ikx}\mathrm{d}x\right)^*$$

So if $f^*(x) = \pm f(x)$ (i.e. $f(x)$ real or imaginary), then $F^*(k) = \pm F(-k)$.

c) Follows from (a) and (b): for $f(x)$ real and symmetric, $F^*(k) = F(-k) = F(k)$, so $F(k)$ is real and symmetric, while if $f(x)$ real and antisymmetric, $F^*(k) = F(-k) = -F(k)$ so $F(k)$ is imaginary and antisymmetric.

d) If $\kappa > 0$ we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\kappa x)\mathrm{e}^{-ikx}\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\mathrm{e}^{-ikx'/\kappa}\frac{1}{\kappa}\mathrm{d}x' = \frac{1}{\kappa}F\left(\frac{k}{\kappa}\right).$$

where the substitution $x' = \kappa x$ has been used. However if $\kappa < 0$ it is slightly more subtle, because when we make the change of variable we interchange the limits: $x = \infty$ corresponds to $x' = -\infty$ and vice versa. So we pick up an overall negative sign, which can be incorporated by replacing the external factor of $1/\kappa$ with $1/|\kappa|$.

e)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+a)\mathrm{e}^{-ikx}\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\mathrm{e}^{-ik(x'-a)}\mathrm{d}x' = \mathrm{e}^{ika}F(k)$$

where $x' = x + a$.

f)

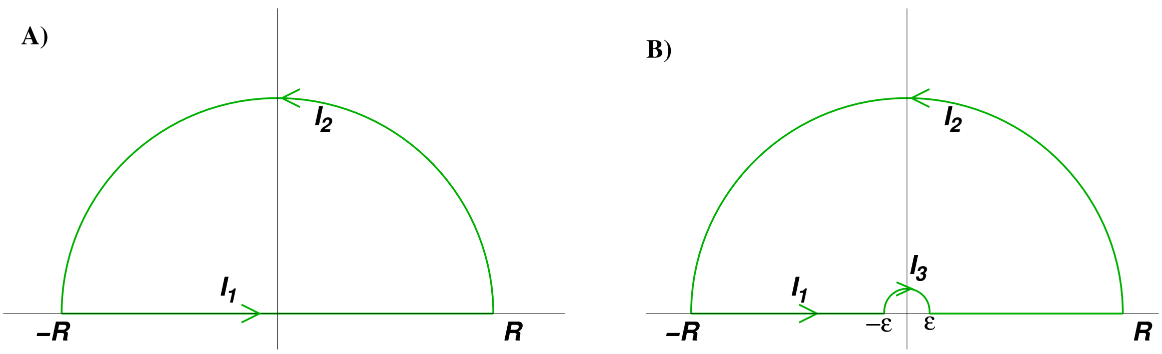
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\alpha x} f(x)\mathrm{e}^{-ikx}\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i(k+i\alpha)x}\mathrm{d}x = F(k + i\alpha).$$

g)
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)\mathrm{e}^{-ikx}\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)\mathrm{e}^{i(-k)x}\mathrm{d}x=f(-k).$$

h)
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-x_0)\mathrm{e}^{-ikx}\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0}$$

i)
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{ik_0x}\mathrm{e}^{-ikx}\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{i(k_0-k)x}\mathrm{d}x=\sqrt{2\pi}\delta(k-k_0)$$

2. Find the Fourier transform of the function defined as $f(x)=\mathrm{e}^{-\kappa x}$ for $x>0$ and $f(x)=0$ for $x<0$. Show also that the inverse transform does restore the original function.



In this question, note that we can write $f(x)=\Theta(x)\mathrm{e}^{-\kappa x}$. The Fourier transform is

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\mathrm{e}^{-\kappa x}\mathrm{e}^{-ikx}\mathrm{d}x=\frac{1}{\sqrt{2\pi}(-\kappa-ik)}\left[\mathrm{e}^{-x(\kappa+ik)}\right]_0^{\infty}=\frac{1}{\sqrt{2\pi}(\kappa+ik)}$$

The inverse transform is $f(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{ikx}\frac{1}{\kappa+ik}\mathrm{d}k.$

To avoid confusion with variables let's do the following integral instead: $I=\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ixt}}{\alpha+ix}\mathrm{d}x=-i\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ixt}}{x-i\alpha}\mathrm{d}x$. Replacing x with z, and considering first $t>0$, we can use the contour of figure (A); Jordan's lemma gives $\lim_{R\to\infty}I_2=0$ and there is one pole in the upper half plane with residue $-i\mathrm{e}^{-\alpha t}$. Hence $I=2\pi\mathrm{e}^{-\alpha t}$.
However if $t<0$, we need to close the contour in the lower half plane in order that the integral round the semicircle at $|z|=R$ vanishes as $R\to\infty$. There are no poles in the lower half plane, so in this case $I=0$.
Hence we can write

$$I\equiv f(t)=2\pi\Theta(t)\mathrm{e}^{-\alpha t}$$

With the variable substitution $x\to k$, $\alpha\to\kappa$ and $t\to x$ we therefore recover $f(x)$.

3. Show the following:
a) The Fourier transform of a "top hat" function of height $1/a$ and width a, centred on x_0, is $\dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0}\mathrm{sinc}(ka/2)$.

b) The Fourier transform of a "triangle" function of height $1/a$ and width $2a$, centred on x_0, is $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0}\mathrm{sinc}^2(ka/4)$. (The function may be written as $\frac{1}{a^2}(a-|x-x_0|)$ for $-a<x<a$.)

c) The Fourier transform of $\frac{1}{\sqrt{2\pi}}\mathrm{sinc}(\kappa(x-x_0))$ is e^{-ikx_0} times a top-hat function of width 2κ and height $1/(2\kappa)$, centred on $k=0$.

(Hint: first use a shift theorem to centre the functions at $x=0$.)

In parts (a) and (b), sketch the functions and comment on the widths of the functions and their transforms.

a)

$$F(k)=\frac{1}{a\sqrt{2\pi}}\int_{x_0-a/2}^{x_0+a/2}\mathrm{e}^{-ikx}\mathrm{d}x=\frac{1}{a\sqrt{2\pi}(-ik)}\Big[\mathrm{e}^{-ikx}\Big]_{x_0-a/2}^{x_0+a/2}=\frac{2}{\sqrt{2\pi}ka}\mathrm{e}^{-ikx_0}\sin(\tfrac{1}{2}ka)$$
$$=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0}\mathrm{sinc}(\tfrac{1}{2}ka)$$

b) We shift the integration variable from x to $x'=x-x_0$ (or just use a shift theorem). We also need to integrate from $-a\to 0$ and from $0\to a$ separately. Suppressing some intermediate lines, we have

$$\begin{aligned}F(k)&=\frac{1}{\sqrt{2\pi}a^2}\int_{x_0-a}^{x_0+a}(a-|x-x_0|)\mathrm{e}^{-ikx}\mathrm{d}x\\&=\frac{\mathrm{e}^{-ikx_0}}{\sqrt{2\pi}a^2}\left(\int_{-a}^{0}(a+x')\mathrm{e}^{-ikx'}\mathrm{d}x'+\int_0^a(a-x')\,\mathrm{e}^{-ikx'}\mathrm{d}x'\right)\\&=\frac{\mathrm{e}^{-ikx_0}}{\sqrt{2\pi}}\frac{2}{a^2k^2}(1-\cos(ak))=\frac{\mathrm{e}^{-ikx_0}}{\sqrt{2\pi}}\mathrm{sinc}^2(\tfrac{1}{2}ka)\end{aligned}$$

c) Since we showed in (a) that the F.T. of a top hat is a sinc function, we could use the result of qu. 1(g) to show immediately that the F.T. of a sinc function is a top hat. However we choose to show it explicitly so that we can use the result later without having applied the inverse transform.

We will need to evaluate integrals of the form $\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\alpha x}}{x}\mathrm{d}x$, and we use contours with a small semicircle in the upper half plane around the origin, and closed with a semi-circle at infinity in the upper half plane if $\alpha>0$ but the lower half plane if $\alpha<0$. (For $\alpha>0$ this is (B) in the diagram above.) With these choices Jordan's lemma ensures the contribution from the semi-circle at infinity vanishes each time. The integral round the small circle is just $-i\pi$. If $\alpha>0$ there is no pole enclosed by the contour, but if $\alpha<0$ the pole, residue 1, is included, contributing $-2i\pi$ to the clockwise contour integral. The result for the integral along the real axis is is

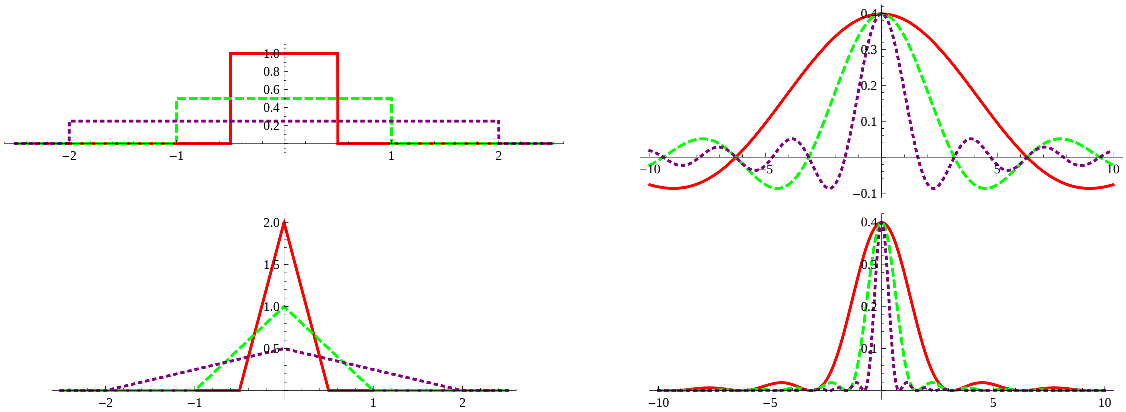
$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\alpha x}}{x}\mathrm{d}x=i\pi\,\mathrm{sgn}(\alpha)\equiv i\pi\frac{|\alpha|}{\alpha}.$$

Now for the I.F.T. of the sinc function, we take $\kappa>0$ (otherwise we replace κ with $|\kappa|$ and pick up an overall minus sign). The shift theorem just gives the result as e^{-ikx_0} times the F.T. of the function with $x_0 = 0$, so we proceed to calculate the latter:

$$F(k) = \text{F.T.}\left[\frac{1}{\sqrt{2\pi}}\text{sinc}(\kappa x)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\text{sinc}(\kappa x)\mathrm{e}^{-ikx}\mathrm{d}x$$
$$= \frac{1}{4i\pi}\int_{-\infty}^{\infty}\frac{1}{\kappa x}\left(\mathrm{e}^{-i(k-\kappa)x} - \mathrm{e}^{-i(k+\kappa)x}\right)\mathrm{d}x$$
$$= \frac{1}{4\kappa}\left(\text{sgn}(k-\kappa) - \text{sgn}(k+\kappa)\right)$$

We distinguish three regions for k: $k < -\kappa$ in which case both sgns give -1 and cancel, $k > \kappa$ where both give 1 and cancel, and $-\kappa < k < \kappa$ where the first gives 1 and the second -1 and they reinforce, to give $F(k) = \dfrac{1}{2\kappa}$ for $-\kappa < k < \kappa$.

Seen as a function of k, this is a top-hat function of width 2κ and height $1/(2\kappa)$, centred on $k = 0$, as required.



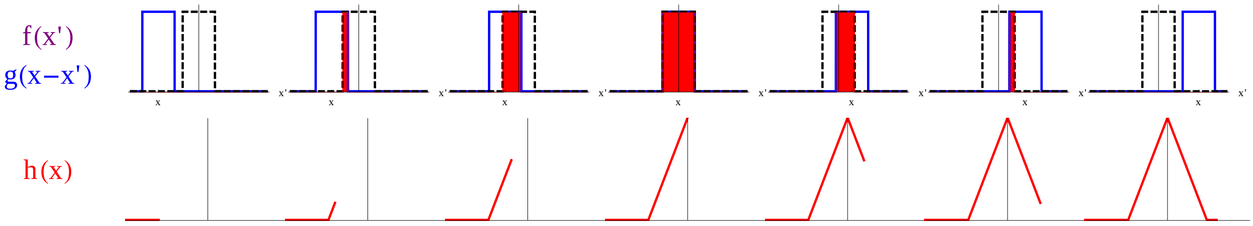
Above we plot the top hat and triangle functions and their Fourier transforms for $a = 1$, 2 and 4. We see that the wider the original function, the narrower the F.T.

4. From the convolution theorem, show that the convolution of two gaussians with width parameters a and b (eg $f(x) = \mathrm{e}^{-x^2/(2a^2)}$) is another with width parameter $\sqrt{a^2+b^2}$.
This convolution can be done directly (which is not what the question asked us to do): if $h = f * g$,

$$h(x) = \int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{2a^2}\right)\exp\left(-\frac{y^2}{2b^2}\right)\mathrm{d}y$$
$$= \exp\left(-\frac{x^2}{2(a^2+b^2)}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{a^2+b^2}{2a^2b^2}\left(y - x\frac{b^2}{(a^2+b^2)}\right)^2\right)\mathrm{d}y$$
$$\propto \exp\left(-\frac{x^2}{2(a^2+b^2)}\right)$$

but a lot of algebra has been omitted in going from the first to the second line. (Working backwards to verify the result is easier.) On the other hand if we recall that the F.T. of $f(x)$ is $\mathrm{e}^{-(ka)^2/2}$ (omitting normalisation constants), we can use the convolution theorem to write the F.T. of the convolution as $\mathrm{e}^{-(ka)^2/2}\mathrm{e}^{-(kb)^2/2} = \mathrm{e}^{-k^2(a^2+b^2)/2}$. And then the inverse F.T. gets us back to $\exp\bigl(-x^2/(2(a^2+b^2))\bigr)$.

5. Show that the triangle function of qu. 3b can be written as a convolution of two identical top-hat functions of half the width. Hence explain the form of the Fourier transform of the triangle function.



The plot above shows on the top line the two top hat functions, one (black, dashed) centred at 0 and the other (blue, solid) with a moving centre x. Let the widths of the top hats be a, and the heights, b. The line below shows the function which is the product of the two. The convolution of the two top hats, as a function of the position of the moving one x, is the area of the overlap (in red), which is plotted as a fucntion of x on the second line. Clearly when $x < -a$, there is no overlap and the convolution is zero. For $-a < x < 0$, the area of overlap grows linearly to reach a maximum at $x = 0$, then for $0 < x_0 < a$ it falls off linearly to reach zero at $x = a$. Hence the convolution is a triangle function with width $2a$.
More formally, for $-a < x < 0$ the region of overlap lies between $-a/2$ and $x + a/2$, giving for the convolution $h(x)$:

$$h(x) = \int_{-a/2}^{x+a/2} b^2 \mathrm{d}x' = b^2(a + x) \qquad -a < x < 0.$$

Similarly for $0 < x < a$ the overlap is non-zero only between $x - a/2$ and $a/2$, giving $h(x) = b^2(a - x)$. Hence $h(x) = b^2(a - |x|)$ for $-a < x < a$, as required.

Since the F.T. of a convolution is a product, the F.T. of a triangle function (convolution of top hats) is the square of the F.T. of a single top hat of half the width. The result of qu. 2(b) thus follows from that of 2(a).

6. Prove the following results for delta functions. In each case except the last, multiply both sides by $f(x)$ and integrate over x (using a shift of variable if required).
   a) $\delta(ax - b) = \dfrac{1}{|a|}\delta(x - b/a)$
   b) $\delta(x^2 - 4) = \frac{1}{4}\big(\delta(x - 2) + \delta(x + 2)\big)$
   c) $\delta\big(g(x)\big) = \sum_i \dfrac{1}{|g'(x_i)|}\delta(x - x_i)$ where x_i are the real (simple) roots of $g(x)$.
   d) $\int_{-\infty}^{x} \delta(x' - a)\mathrm{d}x' = \Theta(x - a)$, where $\Theta(x) = 0$ if $x < 0$ and 1 if $x > 0$.

   a)

$$\int_{-\infty}^{\infty} \delta(ax-b) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \delta(y) f\left(\frac{y+b}{a}\right)\frac{1}{|a|}\mathrm{d}y = \frac{1}{|a|} f\left(\frac{b}{a}\right) = \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(x-b/a) f(x)\mathrm{d}x$$

See solution to qu. 44(d) for the explanation of the modulus sign.

b) We note that x^2-4 vanishes at $x=\pm 2$, so the only contributions to the integral will come from those points:

$$\int_{-\infty}^{\infty} \delta(x^2-4)f(x)\mathrm{d}x = \int_{-2-\epsilon}^{-2+\epsilon} \delta(x^2-4)f(x)\mathrm{d}x + \int_{2-\epsilon}^{2+\epsilon} \delta(x^2-4)f(x)\mathrm{d}x$$

for some small but finite ϵ.

We can find the contributions separately. If we make the variable change $y = x+2$, $x^2-4 = y(y-4) \to -4y$ as $y \to 0$. Thus

$$\int_{-2-\epsilon}^{-2+\epsilon} \delta(x^2-4)f(x)\mathrm{d}x = \int_{-\epsilon}^{\epsilon} \delta(-4y)f(y-2)\mathrm{d}y = \frac{1}{4}f(-2)$$

using the result of (a) in the final step. Similarly the contribution from $x=2$ gives $\frac{1}{4}f(2)$. However the sum of these two contributions is just what we'd get from $\frac{1}{4}\big(\delta(x-2) + \delta(x+2)\big)$.

c) The idea here is the same as the above. The only contributions come from the values of x which are the real zeros of $g(x)$, x_i. We write $y = x - x_i$ and Taylor-expand $g(x)$ to give $g(x_i+y) \approx y g'(x_i)$, so

$$\int_{x_i-\epsilon}^{x_i+\epsilon} \delta\big(g(x)\big)f(x)\mathrm{d}x = \int_{-\epsilon}^{\epsilon} \delta(y g'(x_i))f(x_i+y)\mathrm{d}y = \frac{1}{|g'(x_i)|}f(x_i)$$

The full result is the sum of all such contributions.

d) The integral $\int_{x_1}^{x_2} f(x)\delta(x-x_0)\mathrm{d}x$ is only non-zero if $x_1 < x_0 < x_2$. Hence so long as $x < a$, $\int_{-\infty}^{x} \delta(x'-a)\mathrm{d}x = 0$. However if $x > a$ the integral is 1. This is the same as $\Theta(x)$.

7. Let $f(z)$ be a function which tends to zero as $|z| \to \infty$ and which has only a finite number of simple poles at points z_n in the upper half plane, and none on the real axis, furthermore let $f(x)$ be real on the real axis. Let $b_1^{(n)}$ be the residues of $f(z)$ at these poles. Use the same method that we used in lectures for $\int_{-\infty}^{\infty} \mathrm{sinc} x \,\mathrm{d}x$ to show that

$$\int_{-\infty}^{\infty} \kappa\,\mathrm{sinc}(\kappa x) f(x)\,\mathrm{d}x = \pi f(0) + 2\pi \sum_n \mathrm{Im}\left[i\frac{b_1^{(n)}\exp(i\kappa z_n)}{z_n}\right].$$

Hence show that $\lim_{\kappa\to\infty}\int_{-\infty}^{\infty}(\kappa/\pi)\mathrm{sinc}(\kappa x)f(x)\mathrm{d}x = f(0)$. Thus $\{(\kappa/\pi)\mathrm{sinc}(\kappa x)\}$ is a delta-sequence for functions of this kind. (Can you argue that the restriction to simple poles is not in fact necessary for this final result?)

We will consider $I = \int_{-\infty}^{\infty} \exp(i\kappa x)f(x)/x\,\mathrm{d}x$, noting that $\mathrm{Im}\,[I]$ is the desired integral $\int_{-\infty}^{\infty}\kappa\,\mathrm{sinc}(\kappa x)f(x)\,\mathrm{d}x$. We take a contour like (B) on the previous sheet, with the small semicircle centred on $z=0$ and the large one at a distance large enough to enclose all the pole of $f(z)$ in the UHP. Since all the conditions for Jordan's lemma are satisfied ($\kappa > 0$, finite number of poles, $f(z)/z \to 0$ as $R \to \infty$), $\lim_{R\to\infty} I_2 = 0$. The small circle I_3 picks up minus half of the the residue of the full function $\exp(i\kappa z)f(z)/z$ at $z=0$, which is $f(0)$. The closed contour integral picks up the sum of the residues of the full function at the (simple) poles of $f(z)$ in the UHP, so

$$\begin{aligned} I = \lim_{R\to\infty,\epsilon\to 0} I_1 &= 2\pi i \sum_n \lim_{z\to z_n} (z-z_n)\exp(i\kappa z) f(z)/z - \lim_{R\to\infty,\epsilon\to 0}(I_2+I_3) \\ &= 2\pi i \sum_n \exp(i\kappa z_n)/z_n \lim_{z\to z_n}(z-z_n)f(z) - 0 - \big(-i\pi f(0)\big) \\ &= 2\pi i \sum_n \frac{\exp(i\kappa z_n) b_1^{(n)}}{z_n} + i\pi f(0) \end{aligned}$$

which gives the required result.

The terms $\exp(i\kappa z_n)$ each have a factor $\exp(-\kappa y_n)$, where $y_n > 0$. As $\kappa \to \infty$ these all vanish. Hence $\lim_{\kappa\to\infty} \int_{-\infty}^{\infty} (\kappa/\pi)\mathrm{sinc}(\kappa x) f(x)\mathrm{d}x = f(0)$ and hence $\{(\kappa/\pi)\mathrm{sinc}(\kappa x)\}$ is a delta-sequence here.

The restriction to simple poles can be lifted: if the nth pole has order $p+1$ the residue there will be proportional to $(\mathrm{d}^p/\mathrm{d}z^p)\big((z-z_n)^{(p+1)} f(z)\exp(i\kappa z)/z\big)_{z_n}$. But this will always be proportional to $\exp(i\kappa z_n)$ and hence will also vanish as $\kappa \to \infty$. (The restriction to real functions can also be lifted by replacing $\sin(\kappa x)$ by $e^{i\kappa x} - e^{-i\kappa x}$ and treating the two terms separately, closing the contour in the lower half plane for the second case. Then we need to restrict f to those functions for which Jordan's lemma holds in the lower half plane too.)