PHYS20672 Complex Variables and Integral Transforms: Examples 6

43. Let $f(x)$ be a periodic function with period L. Write down expressions for the Fourier cosine and sine coefficients a_n and b_n. Use the orthogonality on this interval of the complete set of functions $\exp(ik_n x)$ where $k_n = 2\pi n/L$, $n \in \mathbb{Z}$ to show that $f(x)$ can be written

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(ik_n x) \qquad \text{where} \qquad c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\exp(-ik_n x)\mathrm{d}x$$

Show also that $c_0 = a_0$, and $c_n = (a_n - ib_n)/2$, $c_{-n} = (a_n + ib_n)/2$ for $n > 0$. Show this ensures that if $f(x)$ is real, the Fourier series in terms of complex exponentials is also real.

44. Prove the following results for Fourier transforms, where F.T. represents the Fourier transform, and F.T. $[f(x)] = F(k)$:
a) If $f(x)$ is symmetric (or antisymmetric), so is $F(k)$: i.e. if $f(x) = \pm f(-x)$ then $F(k) = \pm F(-k)$.
b) If $f(x)$ is real, $F^*(k) = F(-k)$.
c) If $f(x)$ is real and symmetric (antisymmetric), $F(k)$ is real and symmetric (imaginary and antisymmetric).
d) F.T. $[f(\kappa x)] = \dfrac{1}{|\kappa|}F(\dfrac{k}{\kappa})$.
e) F.T. $[f(x+a)] = \mathrm{e}^{ika}F(k)$.
f) F.T. $[\mathrm{e}^{\alpha x}f(x)] = F(k + i\alpha)$ (for real or complex α.)
g) F.T. $[F(x)] = f(-k)$.
h) F.T. $[\delta(x - x_0)] = \mathrm{e}^{-ikx_0}/\sqrt{2\pi}$
i) F.T. $[\mathrm{e}^{ik_0x}] = \sqrt{2\pi}\delta(k - k_0)$
The results from (e) and (f) are called the shift theorems.

45. Find the Fourier transform of the function defined as $f(x) = \mathrm{e}^{-\kappa x}$ for $x > 0$ and $f(x) = 0$ for $x < 0$. Use the result of qu. 39 with an appropriate change of variable to show that the inverse transform does restore the original function.

46. Show the following:
a) The Fourier transform of a "top hat" function of height $1/a$ and width a, centred on x_0, is $\dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0}\mathrm{sinc}(ka/2)$.
b) The Fourier transform of a "triangle" function of height $2/a$ and width a, centred on x_0, is $\dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-ikx_0}\mathrm{sinc}^2(ka/4)$. (The function may be written as $(2/a - 4|x - x_0|/a^2)$.)

c) The Fourier transform of $\dfrac{1}{\sqrt{2\pi}}\mathrm{sinc}(\kappa(x-x_0))$ is e^{-ikx_0} times a top-hat function of width 2κ and height $1/(2\kappa)$, centred on $k = 0$. (Hint: first use a shift theorem to centre the function at $x = 0$. Write $\sin(\kappa x)$ as a sum of complex exponentials and deal with each part separately, using contour integration and an appropriate contour for integrands with a pole on the real axis (see qu. 38). Whether you close in the upper or lower half plane will depend on the relative sizes of k and κ (see qu. 39).)
In parts (a) and (b), sketch the functions and comment on the widths of the functions and their transforms.

47. From the convolution theorem, show that the convolution of two gaussians with width parameters a and b (eg $f(x) = \mathrm{e}^{-x^2/(2a^2)}$) is another with width parameter $\sqrt{a^2+b^2}$.

48. Show that the triangle function of qu. 46b can be written as a convolution of two identical top-hat functions of half the width. Hence explain the form of the Fourier transform of the triangle function.

49. Prove the following results for delta functions. In each case except the last, multiply both sides by $f(x)$ and integrate over x (using a shift of variable if required).

    a) $\delta(ax-b) = \dfrac{1}{|a|}\delta(x-b/a)$

    b) $\delta(x^2-4) = \frac{1}{4}\big(\delta(x-2)+\delta(x+2)\big)$

    c) $\delta\big(g(x)\big) = \displaystyle\sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i)$ where x_i are the real (simple) roots of $g(x)$.

    d) $\displaystyle\int_{-\infty}^{x} \delta(x'-a)\mathrm{d}x' = \theta(x-a)$, where $\theta(x) = 0$ if $x<0$ and 1 if $x>0$.

50. Let $f(z)$ be a function which tends to zero as $|z| \to \infty$ and which has only a finite number of simple poles at points z_n in the upper half plane, and none on the real axis, furthermore let $f(x)$ be real on the real axis. Let $b_1^{(n)}$ be the residues of $f(z)$ at these poles. Use the same method that we used in lectures for $\int_{-\infty}^{\infty} \mathrm{sinc}x\,\mathrm{d}x$ to show that

$$\int_{-\infty}^{\infty} \kappa\,\mathrm{sinc}(\kappa x) f(x)\,\mathrm{d}x = \pi f(0) + 2\pi \sum_n \mathrm{Im}\left[i\frac{b_1^{(n)}\exp(i\kappa z_n)}{z_n}\right].$$

Hence show that $\lim_{\kappa\to\infty}\int_{-\infty}^{\infty}(\kappa/\pi)\mathrm{sinc}(\kappa x)f(x)\mathrm{d}x = f(0)$. Thus $\{(\kappa/\pi)\mathrm{sinc}(\kappa x)\}$ is a delta-sequence for functions of this kind. (Can you argue that the restriction to simple poles is not in fact necessary for this final result?)

51. Ensure that you can reproduce all the Laplace transforms in the table, particularly those which were not covered in lectures.

52. Find the Laplace transforms of the following functions (using results from tables where appropriate):

$$a)\ t\sin\omega t \qquad b)\ t^{-1}(1-e^{-\alpha t}) \qquad c)\ e^{-\alpha t}t^3\sinh(\beta t)$$

53. Using the table, find the inverse Laplace transforms of the following (you may assume any restrictions on parameters required for the inverse to exist are satisfied):

$$a)\ \frac{s+3}{s(s+1)} \qquad b)\ \frac{1}{(s-1)^2+4} \qquad c)\ \frac{s+1}{s(s^2+1)} \qquad d)\ \frac{\mathrm{e}^{-3s}}{(s-1)^2+4}$$

54. Given that $y(t)$ satisfies $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}+y=t$ with initial conditions $y=1$, $y'=-2$ at $t=0$, show that the Laplace transform of y, $Y(s)$, is

$$\frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$

and hence find $y(t)$.

55. Use Laplace transforms to solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y = \begin{cases} 0 & \text{if } 0 < t < 2, \\ 1 & \text{if } t > 2. \end{cases}$$

subject to the initial conditions $y = 1$, $y' = 2$ at $t = 0$.

56. The functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2x - 3y \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = y - 2x$$

with initial conditions $x = 1$, $y = x' = y' = 0$ at $t = 0$. Solve by using Laplace transforms.

57. Use the convolution theorem to find the following inverse Laplace transform:

$$\mathfrak{L}^{-1}\left(\frac{1}{s^2(s+1)^2}\right) = (t+2)\mathrm{e}^{-t} + t - 2 \qquad \text{for } t > 0.$$

58. Use the Bromwich integral to do the same inverse transform as above.