PHYS20672 Complex Variables and Integral Transforms: Examples 4

28. Using the definitions of the coefficients a_n and b_n in terms of contour integrals, show that the Laurent series of $1/(z+2)$ about $z=1$, for $|z-1|>3$, is

$$\frac{1}{z+2} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{(z-1)^n} = \frac{1}{z-1} - \frac{3}{(z-1)^2} + \frac{9}{(z-1)^3} - \ldots$$

Note that this was derived in lectures using a geometric series.
Hints: The contour of integration C is $|z-1|=R$, for any $R>3$. For a_n the integrand is $\dfrac{1}{(z+2)(z-1)^{n+1}}$ which has a simple pole at $z=-2$ and a pole of order $n+1$ at $z=1$, both within the contour. For b_n the integrand is $\dfrac{(z-1)^{n-1}}{z+2}$ which only has a simple pole at $z=-2$. In each case use the appropriate Cauchy integral formulae to evaluate the integrals, splitting the contour into smaller ones circling each pole as required.

The rest of the questions on this sheet do not require the use of the explicit contour-integral definitions of the coefficients a_n and b_n.

29. Find the first four terms of the Taylor expansion of $\tan z$ about $z=\pi/4$. (Note that $\tan z$ is analytic near $z=\pi/4$, so you can use the usual expression for the coefficients in terms of derivatives of the function.) What is the radius of convergence of this series?

30. If z_0 is a non-zero complex number, find the Taylor series of $f(z)=1/(z-z_0)$ about $z=0$, and show that its radius of convergence is $|z_0|$. (Note that $f(z)$ is analytic in the vicinity of $z=0$.) Check that the result agrees with that obtained using a geometric series.
Use a geometric series to find the Laurent series in the region $|z|>|z_0|$.

31. Use the results above, and partial fractions, to find the Taylor or Laurent series, as appropriate, of $f(z)=\dfrac{z+1}{(z-2)(z-3)}$

    a) about $z=0$, for the region $|z|<2$
    b) about $z=0$, for the region $2<|z|<3$
    c) about $z=0$, for the region $3<|z|$
    d) about $z=2$, for the region $1<|z-2|$ (Hint: change variable to $w=z-2$.)
    e) about $z=1$, for the region $|z-1|<1$
    f) about $z=1$, for the region $1<|z-1|<2$
    g) about $z=1$, for the region $2<|z-1|$.

    (Hint: in (b) we need a Taylor series for the $(z-3)^{-1}$ term, but a Laurent series for the $(z-2)^{-1}$ term. The same idea will apply in some subsequent parts as well.)
    With the help of a spreadsheet or computer program explore numerically the convergence or lack of it of the series (a)–(c), for $z=1$, $z=2.5$ and $z=6$.

32. Find the Laurent series about $z = 0$ of the following functions (you may use standard Taylor series for functions such as e^z where appropriate, and do not need to use explicit contour integration):

$$a)\ \sin(1/z) \qquad b)\ z^{-3}\sin^2 z \qquad c)\ z^3 e^{1/z} \qquad d)\ z^{-2}\,(\cos z - 1)$$

In each case, describe the nature of the singularity. Also, give the value of the coefficient of $1/z$ (b_1, the "residue").

33. Evaluate the residues of $f(z) = \dfrac{z^2+1}{z(z-1)^3}$ at each of its singularities.

34. Evaluate the residues at each of the singularities of the following functions:

$$a)\ \frac{z^2+z-2}{(z-1)^2} \qquad b)\ \frac{1}{\cos z} \qquad c)\ \frac{z}{\sin^2 z} \qquad d)\ \frac{1}{\sin z - \cos z} \qquad e)\ \frac{1}{\sinh z}$$

(Note that $\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b$.)