PHYS20672 Complex Variables and Integral Transforms: Examples 3

20. In each of the following cases evaluate $\int_C f(z)\,\mathrm{d}z$ for the curves C_1 and C_2, where the endpoints are $a = 1$ and $b = i$, and C_1 is the path which follows the axes and passes through the origin, while C_2 is a the straight lines segment $y = 1 - x$.

$$a)\ f(z) = \mathrm{Re}\,(z) \qquad b)\ f(z) = z$$

21. Evaluate $\int_C |z|\,\mathrm{d}z$ for the curves C_1 and C_2, where the endpoints are $a = 1$ and $b = -1$, and C_1 is along the x-axis and C_2 is a semicircle of unit radius in the upper half plane.

22. Writing $z = a + R\mathrm{e}^{i\theta}$, $0 \le \theta \le 2\pi$, and using the path $|z - a| = R$, show

$$a)\ \int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i \qquad b)\ \int_C \frac{1}{(z-a)^n}\,\mathrm{d}z = 0 \quad \text{for integer } n > 1$$

Hence using Cauchy's theorem and partial fractions, find $\int_C f(z)\,\mathrm{d}z$ in the following cases:

(a) $f(z) = 1/(z - i)$; $C : |z| = R$, where i) $R = 1/2$, ii) $R = 2$.
(b) $f(z) = 1/(z^2 - 3z + 2)$; $C : |z| = R$, where i) $R = 1/2$, ii) $R = 3/2$, iii) $R = 5/2$.
(c) $f(z) = (z + 1)/(z^2 - 3z + 2)$ for the same contours as (b).
(d) $f(z) = (z^2 + z + 1)/(z^3 - z^2)$ for the same contours as (a).

23. Use the appropriate Cauchy integral formula to evaluate the following, where C_1 is a circle with $|z| = 1$ and C_2 is a square with corners at $\pm 2, \pm 2 + 4i$.

$$a)\ \oint_{C_1} \frac{e^{3z}}{z}\mathrm{d}z \quad b)\ \oint_{C_1} \frac{\cos^2(2z)}{z^2}\mathrm{d}z \quad c)\ \oint_{C_1} \frac{\sin^2(2z)}{z^2}\mathrm{d}z \quad d)\ \oint_{C_2} \frac{z^2}{z-2i}\mathrm{d}z \quad e)\ \oint_{C_2} \frac{z^2}{z^2+4}\mathrm{d}z$$

24. Show that

$$\left|\frac{1}{z^2+1}\right| \le \frac{1}{R^2-1} \qquad \text{for } |z| = R > 1.$$

(See question 4.) Hence use the estimation lemma to show that

$$\lim_{R\to\infty} \oint \frac{1}{z^2+1}\mathrm{d}z = 0 \qquad \text{for the circular path } |z| = R.$$

Prove the result from Cauchy's integral formula.

25. By writing $z = \mathrm{e}^{i\theta}$ and hence $\mathrm{d}\theta = \mathrm{d}z/(iz)$, and using formulae such as $\cos\theta = \frac{1}{2}(z + z^{-1})$, convert the following to contour integrals around the unit circle and evaluate using the appropriate Cauchy integral formulae:

$$a)\ \int_0^{2\pi} \cos^4\theta\,\mathrm{d}\theta \quad b)\ \int_0^{2\pi} \sin^6\theta\,\mathrm{d}\theta \quad c)\ \int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta \quad d)\ \int_0^{2\pi} \frac{\cos\theta}{4\cos\theta - 5}\,\mathrm{d}\theta$$

$$e)\ \int_0^{2\pi} \frac{\cos 2\theta}{3\cos\theta + 5}\,\mathrm{d}\theta$$

In (c), you should be able to express your answer as $2\pi(2n-1)!!/(2n)!!$, where, eg, $7!! = 7.5.3.1$ and $8!! = 8.6.4.2$.

26. In this question we prove the Cauchy integral formula for $f^{(n)}(a)$ by induction. Start by assuming it holds for $f^{(n-1)}(a)$, and use it in the expression

$$f^{(n)}(a) = \lim_{h \to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h}$$

to show that it then holds for $f^{(n)}(a)$ as well. (This follows the proof in lectures for $f'(a)$.) But since it holds for $n = 1$, it must hold for any positive integer n. If the general case is too hard, start with $f''(a)$ as a warm-up.

27. Verify that the argument theorem holds for the function $f(z) = (2z+1)/(z^2+z-6)$ and the contour $|z| = 5/2$.