PHYS20672 Complex Variables and Integral Transforms: Examples 1

1. Write the following expressions in the form $x+iy$ and sketch their locations on the complex plane:

$$a)\ \tfrac{1}{10}(3+4i)^2 \qquad b)\ \frac{4+5i}{3-4i} \qquad c)\ 3e^{5i\pi/3} \qquad d)\ e^{1+i\pi/3} \qquad e)\ 1+ie^{7i\pi/6}$$

For each case find $|z|$ and the principle value of the argument θ (for $0 \leq \theta < 2\pi$).

2. Sketch the curves

$$a)\ |z-1| = 2 \qquad b)\ \arg(z-i) = \pi/4$$
$$c)\ \mathrm{Re}\,(z^2) = 3 \text{ for } y > 0 \qquad d)\ \mathrm{Re}\,(e^z) = 1 \text{ for } -\pi/2 < y < \pi/2.$$

3. Sketch the regions

$$a)\ \mathrm{Re}\,(z) > -3 \qquad b)\ 1 < |z-1-i| \leq 2 \qquad c)\ |\arg(z-1-i)| \leq \pi/4 \qquad d)\ |z-1| < |z+1|.$$

4. Give a geometric proof (ie using the vector analogy) that for any two complex numbers z_1 and z_2,

$$|(|z_1| - |z_2|)| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|.$$

Hence show that, on the circle $|z| = R$,

$$a)\ R^2 - 1 \leq |z^2 \pm 1| \leq R^2 + 1 \qquad b)\ \left|\frac{z^2+1}{z^2-1}\right| \leq \frac{R^2+1}{R^2-1}.$$

5. Calculate the following:

$$a)\ \sqrt{1+i} \qquad b)\ \mathrm{Ln}\,(1+i) \qquad c)\ \cos(\pi/4+i) \qquad d)\ \arcsin i$$

6. Verify the following identities:

$$a)\ \sinh(iz) = i\sin z \qquad b)\ \sin(iz) = i\sinh z \qquad c)\ \arcsin(iz) = i\mathrm{arcsinh}\, z$$
$$d)\ \mathrm{arcsinh}\, z = \ln\big(z + \sqrt{1+z^2}\big) \qquad e)\ \mathrm{arctanh}\, z = \tfrac{1}{2}\ln\left(\tfrac{1+z}{1-z}\right)$$

Show that $(\cos z)^2 + (\sin z)^2 = 1$ even for complex z.

7. For each of the following functions, give the domain:

$$a)\ f(z) = \frac{1}{z^2+1} \qquad b)\ f(z) = \frac{z}{z+\overline{z}} \qquad c)\ \frac{1}{|z|^2-1} \qquad d)\ \mathrm{Ln}\,(z)$$

For which is the domain an open, connected set of points of the complex plane?

8. Consider the function $f(z) = z^3+5z^2+2$. Calculate (numerically) $f(z)$ for $z = \exp(in\pi/4)$ for $n = 0-8$. Hence sketch the path traced in the w-plane for $w = f(z)$ as z follows the unit circle, with $0 \leq \theta < 2\pi$. Also, sketch a plot of Arg$[w]$ as a function of θ. Repeat for the function z^3+5z^2+8. Relate your results for the increase in Arg$[w]$ to the number of zeros of each function with a modulus less than one.

9. Reproduce the plots of lines of constant u and v given in the lecture handout for $w = z^2$ and $w = \mathrm{Ln}\, z$;