PHYS20672 Complex Variables and Vector Spaces: Solutions 6, Part 2

56. (i) This is *similar* to Q.55, but here we use the Gram–Schmidt process to construct an orthonormal set of $N-1$ vectors $\{|e_j\rangle, j \neq i\}$ from the vectors $\{|a_j\rangle, j \neq i\}$. We then use these to construct

$$|\breve{a}_i\rangle = C_i\Big(|a_i\rangle - \sum_{j\neq i} |e_j\rangle\langle e_j|a_i\rangle\Big),$$

which is orthogonal to all of the vectors $|e_j\rangle$ and hence to all of the vectors $|a_j\rangle$ with $j \neq i$. The coefficient C_i is chosen so that

$$\langle a_i|\breve{a}_i\rangle = C_i\Big(|a_i|^2 - \sum_{j\neq i}|\langle a_i|e_j\rangle|^2\Big) = 1.$$

[To avoid a possible source of confusion, note that the sets of vectors $\{|e_j\rangle, j \neq i\}$ would be *different* for each i, so this would not, in practice, be an *efficient* process for calculating the set of reciprocal vectors.]

(ii) If $|b\rangle = \sum_j b_j|a_i\rangle$, then

$$\langle\breve{a}_i|b\rangle = \sum_j b_j\langle\breve{a}_i|a_j\rangle = \sum_j b_j\delta_{ij} = b_i\,.$$

Using this (and the definition of the outer product) we find

$$\Big(\sum_i |a_i\rangle\langle\breve{a}_i|\Big)|b\rangle = \sum_i |a_i\rangle\langle\breve{a}_i|b\rangle = \sum_i b_i|a_i\rangle = |b\rangle,$$

so we can see that $\sum_i |a_i\rangle\langle\breve{a}_i|$ has the same effect as $\hat{1}$.

(iii) Using the resolution of unity from part (ii), we find:

$$\hat{A} = \hat{A}\hat{1} = \hat{A}\Big(\sum_i |a_i\rangle\langle\breve{a}_i|\Big) = \sum_i |b_i\rangle\langle\breve{a}_i|$$

and

$$\hat{A} = \hat{1}\hat{A}\hat{1} = \Big(\sum_i |a_i\rangle\langle\breve{a}_i|\Big)\hat{A}\Big(\sum_j |a_j\rangle\langle\breve{a}_j|\Big) = \sum_{i,j} |a_i\rangle\langle\breve{a}_i|\hat{A}|a_j\rangle\langle\breve{a}_j|.$$

(iv) $\hat{A}|a_i\rangle = |b_i\rangle$. If the vectors $\{|b_i\rangle\}$ are linearly independent, we can construct the reciprocal vectors, $\{|\breve{b}_i\rangle\}$. [Of course, if they are *not* independent, the construction fails at this point!] Then $\sum_i |b_i\rangle\langle\breve{b}_i| = \hat{1}$, so that

$$\sum_i |b_i\rangle\langle\breve{b}_i| = \sum_i \hat{A}|a_i\rangle\langle\breve{b}_i| = \hat{A}\,\underbrace{\Big(\sum_i |a_i\rangle\langle\breve{b}_i|\Big)}_{\hat{A}^{-1}} = \hat{1},$$

from which we can read off $\hat{A}^{-1}$.

57. The sum of two linear operators, $\hat{C} = \hat{A} + \hat{B}$, and the product with a scalar, $\hat{D} = \lambda\hat{A}$, were both defined in lectures; $\hat{C}$ and $\hat{D}$ are linear operators, so the set is closed under addition and under multiplication by a scalar.

The zero operator was defined by $\hat{0}|a\rangle = |0\rangle$ for all $|a\rangle \in \mathbb{V}$. It is clearly linear, since $\hat{0}(\lambda|b\rangle + \mu|c\rangle)$ and $\lambda\hat{0}|b\rangle + \mu\hat{0}|c\rangle$ both give $|0\rangle$

The additive inverse $(-\hat{A})$ can be defined as $(-1)\hat{A}$. It has the required property

$$\begin{aligned}\big(\hat{A}+(-1)\hat{A}\big)|a\rangle &= \hat{A}|a\rangle + (-1)\hat{A}|a\rangle \\ &= \big(1+(-1)\big)\hat{A}|a\rangle \\ &= 0\big(\hat{A}|a\rangle\big) = |0\rangle\,;\end{aligned}$$

this holds for any $|a\rangle$, so $\big(\hat{A}+(-1)\hat{A}\big) \equiv \hat{0}$. Thus the set of linear operators on $\mathbb{V}$ is a vector space.

58. *The proof follows a similar pattern to the one for Hermitian operators:*

Suppose that $|v_i\rangle$ and $|v_j\rangle$ are eigenvectors of the unitary operator $\hat{U}$,

$$\hat{U}|v_i\rangle = \omega_i|v_i\rangle \quad \text{and} \quad \hat{U}|v_j\rangle = \omega_j|v_j\rangle.$$

We need to investigate the orthogonality (or otherwise) of the eigenvectors, whilst also making use of the property $\hat{U}^\dagger\hat{U} = \hat{1}$. This motivates us to consider

$$\langle v_j|v_i\rangle = \langle v_j|\hat{U}^\dagger\hat{U}|v_i\rangle = \omega_i\langle v_j|\hat{U}^\dagger|v_i\rangle, \tag{1}$$

where we have used the fact that $|v_i\rangle$ is an eigenvector of $\hat{U}$. Now, on the right-hand side of Eq. (1) we have

$$\begin{aligned}\langle v_j|\hat{U}^\dagger|v_i\rangle &= \overline{\langle v_i|\hat{U}|v_j\rangle} \quad \text{(definition of adjoint)} \\ &= \overline{\omega_j\langle v_i|v_j\rangle} \quad \text{(since } |v_j\rangle \text{ is an eigenvector of } \hat{U}) \\ &= \overline{\omega_j}\langle v_j|v_i\rangle.\end{aligned}$$

Inserting this result into (1) we find

$$\langle v_j|v_i\rangle = \omega_i\overline{\omega_j}\langle v_j|v_i\rangle, \quad \text{or} \quad (1-\omega_i\overline{\omega_j})\langle v_j|v_i\rangle = 0. \tag{2}$$

For the last equation to be true, one of the factors $(1-\omega_i\overline{\omega_j})$ and $\langle v_j|v_i\rangle$ must vanish.

If we take $i = j$, the factor $\langle v_i|v_i\rangle \neq 0$, so we can deduce that $|\omega_i|^2 = 1$; thus, $|\omega_i| = 1$. This will hold for every eigenvalue.

If instead we have $i \neq j$ and $\omega_i \neq \omega_j$, then, after multiplying each side by $\overline{\omega_j}$, we find $\omega_i\overline{\omega_j} \neq |\omega_j|^2$; but $|\omega_j|^2 = 1$, so the factor $(1-\omega_i\overline{\omega_j})$ in (2) cannot be zero. Thus, $\langle v_j|v_i\rangle$ must be zero.

59. **Checks of unitarity and Hermiticity:** don't attempt to do them 'by inspection'!

(i) We have

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}^\dagger = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \Rightarrow \text{Hermitian}\,;$$
$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}^\dagger\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \Rightarrow \text{unitary.}$$

An equivalent test of unitarity is to check whether or not the columns are mutually orthogonal unit vectors.

(ii) Same procedure:

$$\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}^\dagger = \begin{pmatrix}0 & 1\\ -i & 0\end{pmatrix} \neq \begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix} \quad \Rightarrow \text{not Hermitian}\,;$$
$$\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}^\dagger\begin{pmatrix}0 & 1\\ -i & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -i & 0\end{pmatrix}\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \Rightarrow \text{unitary}\,.$$

(iii) Hermitian, but <u>not</u> unitary (the columns are not orthogonal unit vectors).

(iv) Neither.

Eigenvalues and eigenvectors: the procedure should be familiar from first year, so we give the details in only two cases.

(i) Eigenvalue equation: $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mu \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, or $\begin{pmatrix} -\mu & -i \\ i & -\mu \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Setting the determinant of the matrix of coefficients to zero gives $\mu^2 - 1 = 0$, so the eigenvalue are $\mu_1 = 1$ and $\mu_2 = -1$. As expected for an Hermitian matrix, the eigenvalues are real; and, as expected for a unitary matrix, they have unit modulus.

To find the eigenvectors, substitute μ_1 and μ_2 into the eigenvalue equation:

E.g., for $\mu_1 = 1$, the first row of $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ tells us that $-u_1 - iu_2 = 0$, or $u_2 = iu_1$. [The second row gives a useful check: $iu_1 - u_2 = 0$, which is the same result.]

Hence, after normalization, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.

[When comparing with your own answers, remember that the overall phase of an eigenvector is not relevant, so that a result that is different by (say) a factor of i is still correct.]

Similarly, for eigenvalue $\mu_2 = -1$, a normalized eigenvector is $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Orthogonality: $\frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}^\dagger \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}\left(1 + (-i)^2\right) = 0$, as expected.

(ii) The eigenvalues satisfy $\mu^2 - i = 0$, so $\mu_{1,2} = \pm(1+i)/\sqrt{2} = \pm e^{i\pi/4}$. They have unit modulus, as expected for the eigenvalues of a unitary matrix.

The corresponding eigenvectors are $\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm e^{-i\pi/4} \end{pmatrix}$. They are orthogonal, as expected for eigenvectors of a unitary matrix.

(iii) The characteristic equation is $\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0$, so $\mu_1 = 1$ and $\mu_2 = 4$.

The eigenvectors are $\dfrac{1}{\sqrt{3}}\begin{pmatrix} 1+i \\ -1 \end{pmatrix}$ for $\mu = 1$ and $\dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ for $\mu = 4$. They are orthogonal, as expected for eigenvectors of an Hermitian matrix.

(iv) The characteristic equation is $(a - \mu)^2 = 0$, so the root $\mu = a$ occurs <u>twice</u>. Inserting $\mu = a$ into the eigenvalue equation we get

$$\begin{pmatrix} a-a & b \\ 0 & a-a \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} bu_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $b \neq 0$, this requires $u_2 = 0$; u_1 can be anything other than zero, so take $u_1 = 1$.

Thus, there is only <u>one</u> eigenvector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

<u>Comments</u>: The matrix is neither Hermitian nor unitary, so there is no reason to expect there to be two linearly independent eigenvectors corresponding to the repeated eigenvalue — though there might have been. In this example there was only one. However, it is still true that the sum of the roots, $a + a = 2a$, equals the trace of the matrix, and their product, a^2, equals the determinant.

60. **Eigenvalues:** With $c \equiv \cos\theta$ and $s \equiv \sin\theta$, the eigenvalue equation is

$$\begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \omega \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The eigenvalue ω satisfies the characteristic equation

$$(1-\omega)\big((c-\omega)^2+s^2\big)=0\,,$$

whose solutions are $\omega_1=1$, $\omega_2=c+is=e^{i\theta}$ and $\omega_3=e^{-i\theta}$.

The matrix is real and orthogonal, which is a special case of unitary; as expected, the eigenvalues all have unit modulus.

Eigenvectors: We assume below that $\theta\neq 0$ or π.

$\underline{\omega_1=1}$: A normalized eigenvector is $\mathbf{v}_1=\begin{pmatrix}0\\0\\1\end{pmatrix}$, either by inspection or by noting that the rotation through angle θ leaves the z axis invariant.

$\underline{\omega_2=e^{i\theta}}$:

$$\begin{pmatrix}-is & -s & 0\\ s & -is & 0\\ 0 & 0 & 1-c-is\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

From the third row we find $z=0$. From either the first or second row we find $y=-ix$, so a normalized eigenvector is $\mathbf{v}_2=\dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix}$.

$\underline{\omega_3=e^{-i\theta}}$: Similarly, $\mathbf{v}_3=\dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\i\\0\end{pmatrix}$.

It is easy to see that ${\mathbf{v}_1}^\dagger\mathbf{v}_2={\mathbf{v}_1}^\dagger\mathbf{v}_3=0$ and that ${\mathbf{v}_2}^\dagger\mathbf{v}_3=\frac{1}{2}\begin{pmatrix}1 & i & 0\end{pmatrix}\begin{pmatrix}1\\i\\0\end{pmatrix}=\frac{1}{2}(1-1+0)=0$.

61. (a) Q.56 showed how to construct the identity operator $\hat{1}$ using any linearly independent set of vectors and their reciprocals. Here we take those vectors to be the eigenvectors of $\hat{A}$. We have

$$\hat{A}=\hat{A}\hat{1}=\hat{A}\Big(\sum_i|u_i\rangle\langle\breve{u}_i|\Big)=\sum_i\hat{A}|u_i\rangle\langle\breve{u}_i|=\sum_i\lambda_i|u_i\rangle\langle\breve{u}_i|.$$

(b) Left eigenvectors: Using the representation of $\hat{A}$ derived in part (a),

$$\begin{aligned}\langle\breve{u}_j|\hat{A}&=\langle\breve{u}_j|\Big(\sum_i\lambda_i|u_i\rangle\langle\breve{u}_i|\Big)\\&=\sum_i\lambda_i\langle\breve{u}_j|u_i\rangle\langle\breve{u}_i|\\&=\sum_i\lambda_i\,\delta_{ji}\langle\breve{u}_i|=\lambda_j\langle\breve{u}_j|\,;\end{aligned}$$

the factor δ_{ji} picks out the term $i=j$ from the sum.

(c) ‡‡ The matrix of eigenvectors, $\mathbf{S}$, has elements $S_{jk}=\langle e_j|u_k\rangle$. We can verify that its inverse has elements ${S^{-1}}_{ij}=\langle\breve{u}_i|e_j\rangle$:

$$\sum_j {S^{-1}}_{ij}S_{jk}=\sum_j\langle\breve{u}_i|e_j\rangle\langle e_j|u_k\rangle=\langle\breve{u}_i|\hat{1}|u_k\rangle=\delta_{ik}\,,$$

in which we spotted a resolution of unity and made use of the definition of the reciprocal vectors to get the final δ_{ik}. We notice that row i of matrix $\mathbf{S}^{-1}$ is the row-vector representing $\langle\breve{u}_i|$; using the terminology of part (b), it is a <u>left</u> eigenvector of matrix $\mathbf{A}$.

If $\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, its elements are

$$\begin{aligned}
A'_{ij} &= \sum_{k,l} \langle \breve{u}_i | e_k \rangle A_{kl} \langle e_l | u_j \rangle \\
&= \sum_{k,l} \langle \breve{u}_i | e_k \rangle \langle e_k | \hat{A} | e_l \rangle \langle e_l | u_j \rangle \\
&= \langle \breve{u}_i | \hat{1}\hat{A}\hat{1} | u_j \rangle = \langle \breve{u}_i | \hat{A} | u_j \rangle = \langle \breve{u}_i | \lambda_j | u_j \rangle = \lambda_j \delta_{ij}\,;
\end{aligned}$$

i.e., the matrix $\mathbf{A}'$ has elements λ_j along the diagonal and is zero elsewhere.

62. The method was sketched in the question. We have $\det(\mathbf{U}^\dagger) = \overline{\det(\mathbf{U}^\mathsf{T})} = \overline{\det \mathbf{U}}$. For any two $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\det(\mathbf{A}\mathbf{B}) = \det \mathbf{A} \det \mathbf{B}$. Then, using $\mathbf{I} = \mathbf{U}^\dagger\mathbf{U}$, we have $\det \mathbf{I} = 1$ and hence $\det(\mathbf{U}^\dagger\mathbf{U}) = \overline{\det \mathbf{U}} \det \mathbf{U} = |\det \mathbf{U}|^2 = 1$.

    This result also follows from the fact that $\det \mathbf{U}$ equals the product of its eigenvalues, all of which have unit modulus in the case of a unitary matrix; see Q.58.

63. A unitary transformation does not alter the determinant or the trace of a matrix:

$$\begin{aligned}
\det(\mathbf{U}^\dagger\mathbf{M}\mathbf{U}) &= \det(\mathbf{U}^\dagger)\,\det \mathbf{M}\,\det \mathbf{U} \\
&= |\det \mathbf{U}|^2 \det \mathbf{M} \\
&= \det \mathbf{M} \quad (\text{see Q.62 for } |\det \mathbf{U}|^2 = 1);
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tr}(\mathbf{U}^\dagger\mathbf{M}\mathbf{U}) &= \mathrm{Tr}(\mathbf{M}\mathbf{U}\mathbf{U}^\dagger) && (\text{cyclic property of the trace}) \\
&= \mathrm{Tr}(\mathbf{M}\,\mathbf{I}) && (\text{since } \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}) \\
&= \mathrm{Tr}\,\mathbf{M}.
\end{aligned}$$

Now, in a representation in which $\mathbf{M}$ is diagonal,

$$\mathbf{M}^{\mathrm{diag}} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix} \quad \text{and} \quad \exp\big(\mathbf{M}^{\mathrm{diag}}\big) = \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_N} \end{pmatrix},$$

so that

$$\det[\exp(\mathbf{M}^{\mathrm{diag}})] = e^{\lambda_1}\,e^{\lambda_2}\,\cdots\,e^{\lambda_N} = e^{\lambda_1+\lambda_2+\ldots+\lambda_N} = \exp[\mathrm{Tr}(\mathbf{M}^{\mathrm{diag}})].$$

But the determinant and trace are unaffected by the unitary transformation, so $\det[\exp \mathbf{M}] = \exp[\mathrm{Tr}\,\mathbf{M}]$.

‡‡ The argument is easily extended to any matrix that can be diagonalized by a similarity transformation, $\mathbf{M}^{\mathrm{diag}} = \mathbf{S}^{-1}\mathbf{M}\mathbf{S}$. As before, the trace and determinant are unaltered by diagonalization:

$$\mathrm{Tr}(\mathbf{M}^{\mathrm{diag}}) = \mathrm{Tr}(\mathbf{S}^{-1}\mathbf{M}\mathbf{S}) = \mathrm{Tr}(\mathbf{M}\mathbf{S}\mathbf{S}^{-1}) = \mathrm{Tr}(\mathbf{M}\mathbf{I}) = \mathrm{Tr}\,\mathbf{M}$$

and

$$\det(\mathbf{M}^{\mathrm{diag}}) = \det(\mathbf{S}^{-1})\,\det \mathbf{M}\,\det \mathbf{S} = \det \mathbf{M},$$

since $\det(\mathbf{S}^{-1}) = 1/\det \mathbf{S}$.

[‡‡‡‡ *Aside*: What if we don't know that the matrix can be diagonalized? In the general case one can consider the function $f(t) = \det(\exp[t\mathbf{M}])$, which satisfies the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = (\mathrm{Tr}\,\mathbf{M})f\,.$$

If you want to derive this, first show that $f(t + \delta t) = f(\delta t)f(t)$; then note that $f(\delta t) = \det[\mathbf{I} + \delta t\,\mathbf{M} + \mathcal{O}(\delta t^2)]$ and expand the determinant to obtain $f(\delta t) = 1 + (\operatorname{Tr}\mathbf{M})\delta t + \mathcal{O}(\delta t^2)$.

The differential equation can be integrated, using the boundary condition $f(0) = 1$, to give

$$f(t) = \det[\exp(t\mathbf{M})] = e^{t\operatorname{Tr}\mathbf{M}}\,.$$

Setting $t = 1$ gives the required formula.]

64. We define

$$D(\lambda) \equiv \sum_{r=0}^{N} c_r \lambda^r,$$

and the corresponding matrix

$$D(\mathbf{M}) \equiv \sum_{r=0}^{N} c_r \mathbf{M}^r, \quad \text{where } \mathbf{M}^0 \equiv \mathbf{I}.$$

If $\mathbf{M}$ is Hermitian or unitary, it can be diagonalized by a unitary transformation $\mathbf{M}^{\text{diag}} = \mathbf{E}^\dagger\mathbf{M}\mathbf{E}$, where the columns of $\mathbf{E}$ are the eigenvectors of $\mathbf{M}$. We can diagonalize $D(\mathbf{M})$ by applying the same transformation. This should be obvious to you from the fact that an eigenvector of $\mathbf{M}$ is also an eigenvector of $D(\mathbf{M})$; alternatively, we can perform the transformation explicitly on each of the terms in $D(\mathbf{M})$. For example, $\mathbf{M}^r$ becomes

$$\begin{aligned}
\mathbf{E}^\dagger\mathbf{M}^r\mathbf{E} &= \mathbf{E}^\dagger\mathbf{M}\mathbf{I}\mathbf{M}\mathbf{I}\ldots\mathbf{I}\mathbf{M}\mathbf{E} && \text{(the factors of } \mathbf{I} \text{ change nothing)}\\
&= \mathbf{E}^\dagger\mathbf{M}\mathbf{E}\mathbf{E}^\dagger\mathbf{M}\mathbf{E}\ldots\mathbf{E}^\dagger\mathbf{M}\mathbf{E} && \text{(using } \mathbf{E}\mathbf{E}^\dagger = \mathbf{I})\\
&= (\mathbf{E}^\dagger\mathbf{M}\mathbf{E})^r = (\mathbf{M}^{\text{diag}})^r,
\end{aligned}$$

which is diagonal. The same will be true for $D(\mathbf{M})$, which is a linear combination of powers of $\mathbf{M}$. Thus,

$$\mathbf{E}^\dagger D(\mathbf{M})\mathbf{E} = D(\mathbf{M}^{\text{diag}}) = \begin{pmatrix} D(\lambda_1) & 0 & \ldots & 0 \\ 0 & D(\lambda_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D(\lambda_N) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \equiv \mathbf{0}\,; \tag{3}$$

the right-hand side is zero because $D(\lambda_i) = 0$ for each eigenvalue λ_i.

Finally, we apply the inverse transformation to each side of (3) to obtain

$$D(\mathbf{M}) = \mathbf{E}\mathbf{0}\mathbf{E}^\dagger = \mathbf{0},$$

which is the required result.

‡‡ For any matrix that can be diagonalized by a similarity transformation $\mathbf{M}^{\text{diag}} = \mathbf{S}^{-1}\mathbf{M}\mathbf{S}$, the same argument goes through with $\mathbf{E}$ replaced by $\mathbf{S}$ and $\mathbf{E}^\dagger$ replaced by $\mathbf{S}^{-1}$.