PHYS20672 Complex Variables and Vector Spaces:
Solutions 6, Part 1

44. Closure under addition:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \in \mathbb{C}^n.$$

Addition of complex numbers is commutative and associative, so addition of these vectors in $\mathbb{C}^n$ will have the same properties.

Closure under multiplication by a scalar λ (i.e., a complex number):

$$\lambda(a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) \in \mathbb{C}^n.$$

Again, associativity, commutativity, and distributivity are inherited from the complex numbers.

The zero vector: $|0\rangle = (0, 0, \ldots, 0) \in \mathbb{C}^n$

Additive inverse: If $|a\rangle = (a_1, a_2, \ldots, a_n)$, then $|-a\rangle = (-a_1, -a_2, \ldots, -a_n)$ is the vector in $\mathbb{C}^n$ that satisfies $|a\rangle + |-a\rangle = |0\rangle$.

It should be clear that the process of verification is usually very simple, so we abbreviate it even further below.

45. (i) If $f_1(x)$ and $f_2(x)$ are real functions of x that satisfy $f_1(0) = f_1(1) = f_2(0) = f_2(1) = 0$, then for real λ and μ, the function $g(x) = \lambda f_1(x) + \mu f_2(x)$ also satisfies the boundary conditions. This verifies closure under vector addition and under multiplication by a scalar.

(ii) Similarly, if $f_1(0) = f_1(1)$ and $f_2(0) = f_2(1)$, then $g(x) = \lambda f_1(x) + \mu f_2(x)$ satisfies $g(0) = g(1)$.

(iii) However, if we impose the condition $f_1(0) = f_2(0) = 1$, we lose the properties of closure under vector addition and under multiplication by a scalar. For example, if $g(x) = f_1(x) + f_2(x)$, then $g(0) = 2$, which violates the boundary condition. Thus, we no longer have a vector space.

46. If $\alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle = |0\rangle$, then

$$(2\alpha, 3\alpha + \beta, -\alpha + 2\beta - 5\gamma) = (0, 0, 0).$$

Thus $2\alpha = 0$, so $\alpha = 0$. Then $3\alpha + \beta = 0$ implies $\beta = 0$ (since $\alpha = 0$). Then $-\alpha + 2\beta - 5\gamma = 0$ implies $\gamma = 0$ (since $\alpha = \beta = 0$).

So the only solution of $\alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle = |0\rangle$ is $\alpha = \beta = \gamma = 0$: $|a\rangle$, $|b\rangle$ and $|c\rangle$ are therefore linearly independent.

If $\alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle = (2, -3, 1)$, then $2\alpha = 2$, $3\alpha + \beta = -3$ and $-\alpha + 2\beta - 5\gamma = 1$. Solving these equations in turn we find $\alpha = 1$, $\beta = -6$ and $\gamma = -\frac{14}{5}$.

47. If $a(X) = a_0 + a_1X + a_2X^2 + a_3X^3$ and $b(X) = b_0 + b_1X + b_2X^2 + b_3X^3$, then $\lambda a(X) + \mu b(X) = (\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)X + (\lambda a_2 + \mu b_2)X^2 + (\lambda a_3 + \mu b_3)X^2$ is also a polynomial of degree not exceeding three, so the set is closed under addition and under multiplication by a scalar.

The 'polynomial' 0 correctly satisfies $a(x) + 0 = 0 + a(x) = a(x)$.

The additive inverse of $1 + iX + (2 + 3i)X^3$ can be obtained by reversing the signs of the coefficients: $-1 - iX - (2 + 3i)X^3$.

Any polynomial of degree up to 3 is a linear combination of the monomials 1, X, X^2 and X^3, all of which belong to the set, and none of which can be expressed as a linear combination of the others. Thus, $\{1, X, X^2, X^3\}$ is a possible basis. The dimension of the vector space is 4.

The case of cubics: The set of cubics is not closed under addition. For example, take $a(X) = 1 + X + X^3$ and $b(X) = X + X^2 - X^3$. Then $a(X) + b(X) = 1 + 2X + X^2$ is not a cubic in X.

48. It is easy to verify that the closure axioms are satisfied, since adding two rational numbers gives a rational number, and multiplying two rationals also gives a rational.

We note that the real number 0 leaves every vector unchanged under addition, so it is a zero vector.

If $x = p + q\sqrt{2}$, with rational p and q, then $-x \equiv (-p) + (-q)\sqrt{2}$ is also in the set, and it satisfies $x + (-x) = 0$.

Basis vectors: A plausible basis is $\{1, \sqrt{2}\}$, suggesting that the dimension of the vector space is 2, but we should check that 1 and $\sqrt{2}$ are linearly independent. First we note that if

$$\alpha 1 + \beta\sqrt{2} = 0, \tag{1}$$

with nonzero coefficients α and β, then $-\alpha/\beta = \sqrt{2}$. But the last equation is an impossibility, if α and β are both rational, because $-\alpha/\beta$ is rational while $\sqrt{2}$ is irrational. Thus, the only solution of Eq. (1) is $\alpha = \beta = 0$, which shows that 1 and $\sqrt{2}$ are linearly independent.

Uniqueness of the zero vector: Suppose that there is a second zero vector, $0' = \alpha 1 + \beta\sqrt{2}$. For any $x = p + q\sqrt{2}$ it needs to satisfy the equation

$$x + 0' = x, \quad \text{or} \quad (p + \alpha)1 + (q + \beta)\sqrt{2} = p1 + q\sqrt{2}.$$

We have shown that 1 and $\sqrt{2}$ are linearly independent, so the last equation requires $p + \alpha = p$ and $q + \beta = q$; i.e., it requires $\alpha = \beta = 0$. Thus, $0'$ is identical to 0.

49. (i) The equation $|a\rangle + |0'\rangle = |a\rangle$ must hold, in particular, for $|a\rangle = |0\rangle$:

$$|0\rangle + |0'\rangle = |0\rangle. \tag{2}$$

But on the left-hand side of Eq. (2), $|0\rangle + |0'\rangle$ can be replaced by $|0'\rangle$ (using the fact that $|0\rangle$ is a zero vector). Thus, (2) becomes

$$|0'\rangle = |0\rangle.$$

(ii) Let the supposed alternative inverse be $|{\sim}a\rangle$. Then

$$|{\sim}a\rangle + |a\rangle = |0\rangle.$$

Adding $|-a\rangle$ to each side (and using the associativity of addition) gives

$$|{\sim}a\rangle + (|a\rangle + |-a\rangle) = |-a\rangle.$$

But $(|a\rangle + |-a\rangle) = |0\rangle$, so we find that $|{\sim}a\rangle = |-a\rangle$.

(iii) From the axioms for multiplication by a scalar,

$$|a\rangle + 0|a\rangle = 1|a\rangle + 0|a\rangle = (1 + 0)|a\rangle = 1|a\rangle = |a\rangle.$$

Adding $|-a\rangle$ to each side (and using $|a\rangle + |-a\rangle = |0\rangle$) then gives $0|a\rangle = |0\rangle$.

50. The results (and the amount of work involved) depend on the order in which we take the vectors.

E.g., if we take

$$\begin{aligned} |a_1\rangle &= -5\mathbf{k} \\ |a_2\rangle &= \mathbf{j} + 2\mathbf{k} \\ |a_3\rangle &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \end{aligned}$$

we first find $|e_1\rangle = |a_1\rangle/|a_1| = -\mathbf{k}$. (Of course, $+\mathbf{k}$ would do just as well. The same goes for the other choices of sign made later on.)

Next we construct a vector orthogonal to $|e_1\rangle$:

$$|e_2\rangle = C_2(|a_2\rangle - |e_1\rangle\langle e_1|a_2\rangle) = C_2\mathbf{j}.$$

Because we want $|e_2\rangle$ to be a unit vector, we choose $C_2 = 1$, which gives $|e_2\rangle = \mathbf{j}$.

Continuing this procedure, we construct a vector orthogonal to $|e_1\rangle$ and $|e_2\rangle$:

$$|e_3\rangle = C_3(|a_3\rangle - |e_1\rangle\langle e_1|a_3\rangle - |e_2\rangle\langle e_2|a_3\rangle) = C_3(2\mathbf{i}).$$

Choosing $C_3 = \frac{1}{2}$ gives us a unit vector: $|e_3\rangle = \mathbf{i}$.

51. (i) If either vector is zero, both sides are zero, so there is nothing to prove: we simply have an equality in this case.

If $|b\rangle \neq |0\rangle$, we can define $|c\rangle = |a\rangle - \lambda|b\rangle$, where $\lambda = \langle b|a\rangle/\langle b|b\rangle$. Then

$$\begin{aligned} \langle c|c\rangle &= \langle c|a\rangle - \lambda\langle c|b\rangle \\ &= \langle a|a\rangle - \overline{\lambda}\langle b|a\rangle - \lambda\langle a|b\rangle + \lambda\overline{\lambda}\langle b|b\rangle \\ &= \langle a|a\rangle - \frac{|\langle a|b\rangle|^2}{\langle b|b\rangle}, \end{aligned}$$

after some simplification in which we mustn't forget that $\langle a|b\rangle = \overline{\langle b|a\rangle}$.

But $\langle c|c\rangle \geq 0$, so we have

$$\langle a|a\rangle - \frac{|\langle a|b\rangle|^2}{\langle b|b\rangle} \geq 0,$$

giving

$$|\langle a|b\rangle|^2 \leq \langle a|a\rangle\langle b|b\rangle = |a|^2|b|^2.$$

On taking the positive square root of each side, we find $|\langle a|b\rangle| \leq |a||b|$.

(ii) If $|c\rangle = |a\rangle + |b\rangle$, then

$$|c|^2 = \langle c|c\rangle = |a|^2 + |b|^2 + \langle a|b\rangle + \langle b|a\rangle.$$

But we can write $\langle a|b\rangle = X + iY$, where $X, Y \in \mathbb{R}$. So

$$\begin{aligned} \langle a|b\rangle + \langle b|a\rangle &= 2X \\ &\leq 2|X| = 2\sqrt{X^2} \\ &\leq 2\sqrt{X^2 + Y^2} \\ &= 2|\langle a|b\rangle| \leq 2|a||b|, \end{aligned}$$

where Schwarz's inequality is used in the last step. Thus

$$|c|^2 \leq |a|^2 + |b|^2 + 2|a||b| = (|a| + |b|)^2\,;$$

taking the positive square root of each side gives $|c| \leq |a| + |b|$.

Last part: Write $|a\rangle = |c\rangle + |-b\rangle$. Then the triangle inequality gives $|a| \leq |c| + |(-b)|$. But the norm of $|-b\rangle$ is identical to the norm of $|b\rangle$, so we have $|a| \leq |c| + |b|$, or $|a| - |b| \leq |c|$. Similarly, we can show that $|b| - |a| \leq |c|$. Of these last two inequalities, one must have a left-hand side that is nonnegative and equal to $|(|a| - |b|)|$. Thus we have

$$|(|a| - |b|)| \leq |c|.$$

52. To reduce the typing, I'll use a representation in which

$$|e_1\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e_2\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$|a\rangle \longrightarrow \begin{pmatrix} 3i \\ -7i \end{pmatrix} \quad \text{and} \quad |b\rangle \longrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

giving

$$\begin{aligned} |a|^2 &= \begin{pmatrix} -3i & 7i \end{pmatrix} \begin{pmatrix} 3i \\ -7i \end{pmatrix} = 9 + 49 = 58, \\ |b|^2 &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 + 4 = 5, \\ \langle a|b\rangle &= \begin{pmatrix} -3i & 7i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 11i. \end{aligned}$$

Then

$$|c\rangle = |a\rangle + |b\rangle \longrightarrow \begin{pmatrix} 1 + 3i \\ 2 - 7i \end{pmatrix},$$

for which

$$|c|^2 = |1 + 3i|^2 + |2 - 7i|^2 = 63.$$

Schwarz's inequality: $|\langle a|b\rangle| = 11$ and $|a||b| = \sqrt{58} \times \sqrt{5} \simeq 17.0$, so that the inequality $|\langle a|b\rangle| \leq |a||b|$ is satisfied.

Triangle inequality: $|c| = \sqrt{63} \simeq 7.94$ and $|a| + |b| = \sqrt{58} + \sqrt{5} \simeq 9.85$. The inequality $|c| \leq |a| + |b|$ is satisfied.

53. (i) To avoid confusion over suffixes, we use j as the summation variable in $|b\rangle = \sum_j b_j |e_j\rangle$. To calculate $\langle e_i|b\rangle$ we use the fact that the inner product is a linear function of its second argument, $|b\rangle$:

$$\langle e_i|b\rangle = \sum_j b_j \langle e_i|e_j\rangle = \sum_j b_j\, \delta_{ij} = b_i\,;$$

in the second sum over j, the presence of δ_{ij} ensures that the only nonzero term is the one with $j = i$.

We can do the same job for $|a\rangle$ to show that $a_i = \langle e_i|a\rangle$. Taking the complex conjugate gives $\overline{a_i} = \overline{\langle e_i|a\rangle} = \langle a|e_i\rangle$.

(ii) We have $\langle a|b\rangle = \sum_i \langle a|e_i\rangle b_i$. But, from the last line of part (i), $\langle a|e_i\rangle = \overline{a_i}$. Hence $\langle a|b\rangle = \sum_i \overline{a_i}\, b_i$.

54. We write the polynomials as

$$\begin{aligned} p_0(x) &= a_0, \\ p_1(x) &= b_0 + b_1 x, \\ p_2(x) &= c_0 + c_1 x + c_2 x^2, \end{aligned}$$

in which the coefficients are all real. Then

$$\begin{aligned}
\langle p_0|p_0\rangle &= \int_{-1}^{1} |a_0|^2 \,\mathrm{d}x = 2a_0^2 = 1 \qquad \Rightarrow a_0 = \frac{1}{\sqrt{2}} \qquad (\text{or } \; -1/\sqrt{2})\,; \\
\langle p_0|p_1\rangle &= \frac{1}{\sqrt{2}} \int_{-1}^{1} (b_0 + b_1 x) \,\mathrm{d}x = \sqrt{2}\, b_0 = 0 \qquad \Rightarrow b_0 = 0\,; \\
\langle p_1|p_1\rangle &= \int_{-1}^{1} (b_1 x)^2 \,\mathrm{d}x = 2b_1^2/3 = 1 \qquad \Rightarrow b_1 = \sqrt{3/2}\,; \\
\langle p_1|p_2\rangle &= \sqrt{\frac{3}{2}} \int_{-1}^{1} (c_0 x + c_1 x^2 + c_2 x^3) \,\mathrm{d}x = \sqrt{\frac{3}{2}} \times \frac{2}{3}\, c_1 = 0 \qquad \Rightarrow c_1 = 0\,; \\
\langle p_0|p_2\rangle &= \frac{1}{\sqrt{2}} \int_{-1}^{1} (c_0 + c_2 x^2) \,\mathrm{d}x = \frac{2}{\sqrt{2}} (c_0 + c_2/3) = 0 \qquad \Rightarrow p_2(x) = c_0(1 - 3x^2)\,; \\
\langle p_2|p_2\rangle &= c_0^2 \int_{-1}^{1} (1 - 3x^2)^2 \,\mathrm{d}x = 8c_0^2/5 = 1 \qquad \Rightarrow c_0 = \sqrt{5/8}\,.
\end{aligned}$$

So the final results are $p_0(x) = \frac{1}{\sqrt{2}}$, $p_1(x) = \sqrt{\frac{3}{2}}\, x$ and $p_2(x) = \sqrt{\frac{5}{8}}\,(1 - 3x^2)$. Apart from the normalization, these are the first three Legendre polynomials.

55. One way to the solve the problem is to use the Gram–Schmidt process to construct an orthonormal basis for $\mathbb{V}^N$.

Starting with $|e_1\rangle = |a\rangle/|a|$, the remaining $N-1$ unit vectors $\{|e_i\rangle, i = 2, 3, \ldots, N\}$ span $\mathbb{W}$: any linear combination $|b\rangle = \sum_{i=2}^{N} b_i |e_i\rangle$ satisfies $\langle a|b\rangle = 0$.