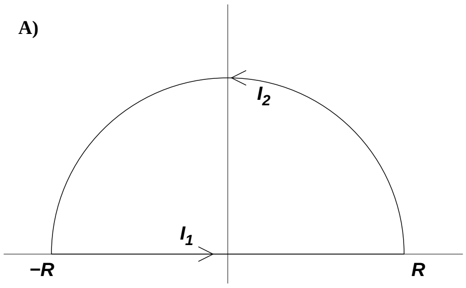
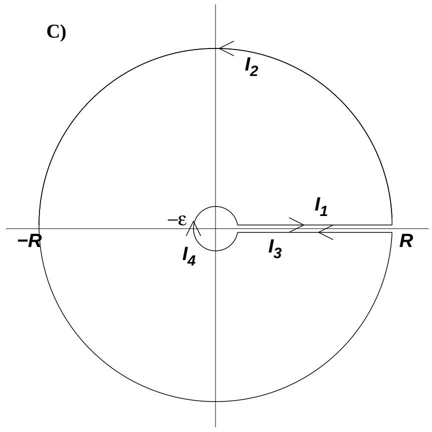
PHYS20672 Complex Variables and Vector Spaces: Solutions 5

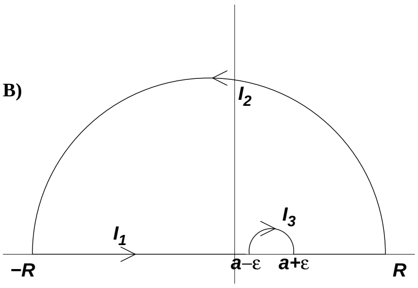
A)



C)



B)



Contours illustrated in figures (A), (B) and (C) are referred to below. *In all questions where real integrals are evaluated using contour integration methods, it is crucial to explain carefully the link between the two.*

36. *See figure (A) above.* In these examples, we denote the integrand by $f(x)$ and the requested real integral by I. We consider the contour integral of $f(z)$ along a contour consisting of the real axis from $z = -R$ to $z = R$ plus the semicircle $|z| = R$ in the upper half plane, and we denote these two contributions to the contour integral as I_1 and I_2. We take R sufficiently large that all poles in the upper half plane are within the contour. As a result the contour integral is independent of R and is determined by the residues at the poles in the upper half plane. Since in all cases $f(z)$ falls off faster than $1/R$, $\lim_{R\to\infty} I_2 = 0$. Furthermore as $f(z) = f(x)$ on the real axis, $\lim_{R\to\infty} I_1 = I$. So $I = 2\pi i$(sum of residues in the upper half plane).

a) $f(z) = \dfrac{1}{1+z^4}$. The poles of $f(z)$ are at the fourth roots of -1, with $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$ lying in the upper half plane. The residues at z_i are

$$b_1^{z=z_i} = \lim_{z\to z_i} \frac{z-z_i}{1+z^4} = \frac{1}{4z_i^3}$$

(using l'Hôpital's rule) so the sum of the two residues in the upper half plane is

$$\frac{1}{4}\left(\frac{1}{e^{3i\pi/4}} + \frac{1}{e^{9i\pi/4}}\right) = \frac{1}{4}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right) = -\frac{i}{2\sqrt{2}}.$$

Thus $I = \pi/\sqrt{2}$.

b) $f(z) = \dfrac{z^4}{1+z^8}$. The poles are at the eighth roots of -1, with $z_1 = e^{i\pi/8}$, $z_2 = e^{3i\pi/8}$, $z_3 = e^{5i\pi/8}$ and $z_4 = e^{7i\pi/8}$ lying in the upper half plane.

The residues at z_i are

$$b_1^{z=z_i} = \lim_{z\to z_i} \frac{(z-z_i)z^4}{1+z^8} = \lim_{z\to z_i} \frac{z^4 + 4(z-z_i)z^3}{8z^7} = \frac{1}{8z_i^3}$$

and the sum of the residues in the upper half plane is $\frac{1}{8}(-2i\sin 3\pi/8 - 2i\sin 9\pi/8) = -(i/2)\sin(12\pi/16)\cos(6\pi/16) = -i\sin\frac{\pi}{8}/(2\sqrt{2})$. (The four terms that go into this sum form two pairs which have cancelling real parts and reinforcing imaginary parts.) Hence $I = \pi\sin(\pi/8)/\sqrt{2}$.

c) $f(z) = \dfrac{1}{(z^2-2z+5)^2} = \dfrac{1}{(z-1-2i)^2(z-1+2i)^2}$.
There is a double pole at $z = 1+2i$, and the residue is

$$\lim_{z\to 1+2i}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{(z-1+2i)^2}\right) = -\left.\frac{2}{(z-1+2i)^3}\right|_{z=1+2i} = \frac{1}{32i}$$

Hence $I = \pi/16$.

37. *See figure (A) above.* The general method is the same as above, but we need to check the conditions of Jordan's lemma in order to say that I_2 vanishes.

a) $I = \displaystyle\int_{-\infty}^{\infty}\frac{x\sin x}{(1+x^2)^2}\,\mathrm{d}x$: we will compute the contour integral of $e^{iz}f(z)$ with $f(z) = z(1+z^2)^{-2}$ and take the imaginary part. We will complete the contour in the upper half plane as the constant in the exponent, 1, is greater than zero. In addition $f(z)$ tends to zero as $|z|\to\infty$, and it has only one (double) pole in the upper half plane, at $z = i$. So the three conditions of Jordan's lemma are satisfied. (The other pole is at $-i$.) The residue at $z = i$ is

$$b_1 = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{ze^{iz}}{(z+i)^2}\right)_{z=i} = \left.\frac{ie^{iz}(z^2+2iz+1)}{(z+i)^3}\right|_{z=i} = \frac{1}{4e}$$

and the desired integral is

$$I = \mathrm{Im}(2\pi i b_1) = \frac{\pi}{2e}.$$

[Note also that $\int_{-\infty}^{\infty} x\cos x/(1+x^2)^2\mathrm{d}x = \mathrm{Re}(2\pi i b_1) = 0$, which is obvious from the symmetry of the integrand.]

b) $I = \displaystyle\int_{-\infty}^{\infty}\frac{\sin\pi x}{1+x+x^2}\,\mathrm{d}x$: we want the imaginary part of the integral of $e^{i\pi z}f(z)$ with $f(z) = (1+z+z^2)^{-1}$. We will complete the contour in the upper half plane as the constant in the exponent, π, is greater than zero. In addition $f(z)$ tends to zero as $|z|\to\infty$, and it has only one pole in the upper half plane, at $z = z_1 = (-1+i\sqrt{3})/2$. So the three conditions of Jordan's lemma are satisfied. (The other pole is at $\overline{z_1}$.) The residue at z_1 is

$$b_1 = \left.\frac{e^{i\pi z}}{z-\overline{z_1}}\right|_{z=z_1} = \frac{e^{-i\pi/2}e^{-\pi\sqrt{3}/2}}{i\sqrt{3}}$$

and the desired integral is

$$I = \mathrm{Im}(2\pi i b_1) = -\frac{2\pi}{\sqrt{3}}e^{-\pi\sqrt{3}/2}$$

This also shows that $\displaystyle\int_{-\infty}^{\infty}\frac{\cos\pi x}{1+x+x^2}\,\mathrm{d}x = \mathrm{Re}(2\pi i b_1) = 0$, which is a consequence of the symmetry about the point $z = -1/2$.

38. We use the substitution $z = a + \epsilon e^{i\theta}$; the contour is like I_3 in (B) above, but traversed anti-clockwise.

$$\lim_{\epsilon\to 0}\int_C (z-a)^n \mathrm{d}z = \lim_{\epsilon\to 0} i\epsilon^{n+1}\int_0^\pi e^{i(n+1)\theta}\mathrm{d}\theta = \lim_{\epsilon\to 0}\frac{\epsilon^{n+1}}{n+1}\big((-1)^{n+1}-1\big) \quad \text{for } n\neq -1$$

If n is odd (and not -1) this vanishes. Furthermore if $n > -1$ the limit as $\epsilon \to 0$ is 0, but if n is even and < -1 it blows up and hence is undefined.

If $n = -1$ we have

$$\lim_{\epsilon\to 0}\int_C \frac{1}{z-a}\mathrm{d}z = \lim_{\epsilon\to 0} i\int_0^\pi \mathrm{d}\theta = \pi i$$

If $f(z)$ has a pole at $z = a$, we can write $f(z) = g(z)/(z-a)$ where $g(z)$ is analytic in the vicinity of $z = a$. Then, by an argument just like the one used to prove Cauchy's integral formula, we have

$$\lim_{\epsilon\to 0}\int_C \frac{g(z)}{z-a}\mathrm{d}z = ig(a)\int_0^\pi \mathrm{d}\theta = \pi i g(a) \qquad \text{where } g(a) = \lim_{z\to a}(z-a)f(z) = b_1^{z=a}$$

a)

$$\lim_{\epsilon\to 0}\int_C \frac{e^z}{z}\mathrm{d}z = \pi i e^0 = \pi i$$

b)

$$\lim_{\epsilon\to 0}\int_C \frac{z^2-2z+1}{z+1}\mathrm{d}z = \pi i(z^2-2z+1)_{z=-1} = 4\pi i$$

c) In this case, although the denominator is z^2 we have a simple pole because $1-e^z$ has a simple zero at $z = 0$:

$$\lim_{\epsilon\to 0}\int_C \frac{1-e^z}{z^2}\mathrm{d}z = \pi i \lim_{z\to 0}\frac{1-e^z}{z} = \pi i \left.\frac{\mathrm{d}(1-e^z)}{\mathrm{d}z}\right|_{z=0} = -i\pi.$$

39. *See figure (B) above.* In these problems, there is one or more poles on the real axis and the contour detours into the upper half plane on small semicircles of radius ϵ to avoid them. For a single pole we will call the integral around this semicircle I_3, with I_1 being used for the sum of the two integrals along the real axis on either side of the pole. Then I is the limit of I_1 as both $R \to \infty$ and $\epsilon \to 0$. If there is more than one pole on the real axis, we need more than one small semicircle, with the contributions being I_4, etc. We calculate the integrals around these semicircles using the results of the previous question, but remembering that we traverse the semicircle in a *clockwise* direction.

a) $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{(x-2)(x^2+1)}\mathrm{d}x$: we take the contour integral of $f(z) = 1/((z-2)(z^2+1))$ around the contour described above, with I_3 being the integral around the small semicircle centred on $z = 2$. There is one pole inside the contour, at $z = i$, and the residue there is

$$b_1^{z=i} = \left.\frac{1}{(z-2)(z+i)}\right|_{z=i} = \frac{1}{2i(i-2)}$$

As $f(z)$ falls off faster than $1/R$ as $R\to\infty$, I_2 will tend to zero. There is a simple pole at $z = 2$ with residue $1/(z^2+1)|_{z=2} = 1/5$, so the limit as $\epsilon \to 0$ of I_3 is $-\pi i/5$. Hence

$$I = \lim_{R\to\infty,\epsilon\to 0} I_1 = 2\pi i b_1^{z=i} - \lim_{R\to\infty,\epsilon\to 0}(I_2+I_3) = \frac{\pi}{(i-2)} + \frac{\pi i}{5} = -\frac{2\pi}{5}$$

b) $I = \displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2-4}\,\mathrm{d}x$: we take the contour integral of $f(z) = \dfrac{e^{iz}}{z^2-4}$ around the contour as described above, except that now there are two poles on the real axis with I_3 being the integral around the small semicircle centred on $z=-2$, and I_4 that centred on $z=2$. There are no poles within the contour, so $I_1+I_2+I_3+I_4=0$. The integrand satisfies the conditions of Jordan's Lemma for the upper half plane, so I_2 will not contribute as $R \to 0$. The residue at $z=2$ is $\left.\dfrac{e^{iz}}{z+2}\right|_{z=2} = e^{2i}/4$ and that at $z=-2$ is $\left.\dfrac{e^{iz}}{z-2}\right|_{z=-2} = -e^{-2i}/4$. Hence

$$I = \lim_{R\to\infty,\epsilon\to 0} I_1 = -\lim_{R\to\infty,\epsilon\to 0}(I_2+I_3+I_4) = i\pi\frac{e^{2i}-e^{-2i}}{4} = -\tfrac{1}{2}\pi\sin 2.$$

c) $I = \displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1-\cos(2x)}{2x^2}\mathrm{d}x$: we take the contour integral of $\dfrac{1-e^{2iz}}{2z^2}$ around the contour described above, with I_3 being the integral around the small semicircle centred on $z=0$. There are no poles within the contour, so $I_1+I_2+I_3=0$. For I_2 we look at the two terms separately; since the conditions for Jordan's lemma hold for e^{2iz}/z^2, and as $1/z^2$ falls off as $1/R^2$, we see that as $R\to\infty$, I_2 will tend to zero. There is a simple pole at $z=0$ with residue $\left.\dfrac{\mathrm{d}(1-e^{2iz})/2}{\mathrm{d}z}\right|_{z=0} = -i$, so the limit as $\epsilon \to 0$ of I_3 is $-\pi$. Hence

$$I = \mathrm{Re}(\lim_{R\to\infty,\epsilon\to 0} I_1) = -\mathrm{Re}\Big(\lim_{R\to\infty,\epsilon\to 0}(I_2+I_3)\Big) = \pi$$

‡‡ *What if we don't want to do a principal-value integral?* As the integrand is analytic for all finite z, the result won't be changed by diverting the integration contour to avoid $z=0$. Suppose that our new contour (call it D) runs from $-\infty$ to ∞, passing below $z=0$. Because $\sin^2 z = \frac{1}{2}(1-\frac{1}{2}[e^{2iz}+e^{-2iz}])$ we can write

$$I = \frac{1}{2}\int_D \frac{1}{z^2}\,\mathrm{d}z - \frac{1}{4}\int_D \frac{e^{2iz}}{z^2}\,\mathrm{d}z - \frac{1}{4}\int_D \frac{e^{-2iz}}{z^2}\,\mathrm{d}z;$$

the point of diverting the contour is that each of the integrals along D is now well defined and can be calculated independently of the other two. The first integral gives $\big[-1/z\big]_{-\infty}^{\infty} = 0$. The third also gives zero: the contour D can be closed by a semicircle of large radius in the LHP, but no poles are enclosed. The second integral must give us the result. In this case D can be completed by a large semicircle in the UHP [Jordan's lemma], so that it encloses the pole of order 2 at $z=0$. The first few terms of the Laurent expansion of e^{2iz}/z^2 are $(z^{-2}+2i/z+\frac{1}{2}(2i)^2+\mathcal{O}(z))$, from which we can read off the residue $b_1 = 2i$. Thus,

$$I = -\frac{1}{4}\int_D \frac{e^{2iz}}{z^2} = -\tfrac{1}{4}\times 2\pi i \times 2i = \pi.$$

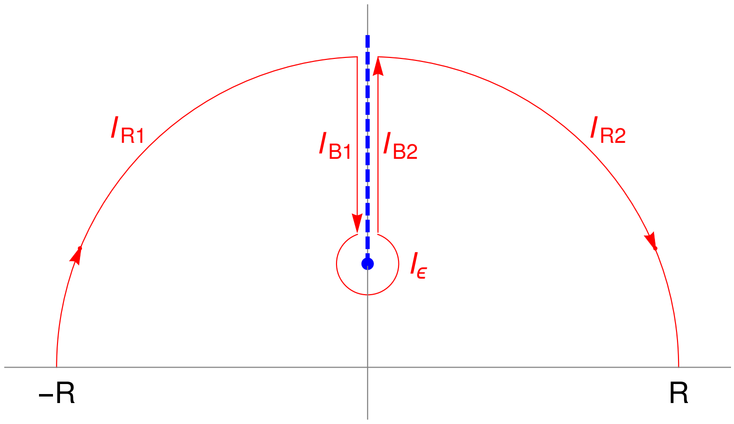
40. (a) $I = \displaystyle\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - i\alpha}\,\mathrm{d}\omega$. Replacing ω with z, and considering first $t>0$, we can use the contour of figure (A); Jordan's lemma gives $\lim_{R\to\infty} I_2 = 0$ and there is one pole in the upper half plane with residue $e^{-\alpha t}$. Hence $I = 2\pi i e^{-\alpha t}$.
However if $t<0$, we need to close the contour in the lower half plane in order that the integral around the semicircle at $|z|=R$ vanishes as $R\to\infty$. There are no poles in the lower half plane, so in this case $I=0$.
Hence we can write

$$I \equiv f(t) = 2\pi i\,\theta(t)e^{-\alpha t} \qquad \text{where } \theta(t) = \begin{cases} 1 & \text{if } t>0, \\ 0 & \text{if } t<0. \end{cases}$$

We might recognize this as the inverse Fourier transform of $1/(\omega - i\alpha)$. Check that the Fourier transform of $f(t)$ above does indeed have this form – but don't fuss over factors of 2π.

(b) $I = \displaystyle\int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{x-ia}}\,\mathrm{d}x$. The integrand has a branch point in the UHP. For $k<0$, the analysis is very similar to that done in part (a): the contour can be completed by a large semicircle of radius R in the LHP; the square-root factor tends to zero for $z \to \infty$; the integrand is meromorphic in the LHP [actually, it is *analytic* there]; thus, Jordan's lemma applies to this case and the integral around the semicircle will tend to zero for $R \to \infty$. Applying the residue theorem gives the result $I = 0$, as no poles are enclosed.

*‡ For $k > 0$, we must do quite a bit more work than in part (a), because the singularity at $z = ia$ is a branch point, rather than a pole. Consistent with the condition $\mathrm{Re}[\sqrt{x-ia}] > 0$ for real x, we can take the branch cut of the square-root function to run along the imaginary axis from ia to $+i\infty$. We deform the path of integration as shown below:



The red line shows the path of integration after it has been pushed up from the real axis into the upper half plane, without crossing the branch cut (shown dashed). The same argument that was used in lectures to prove Jordan's lemma can be used to show that the integrals I_{R1} and I_{R2} along the quarter-circles of radius R in the UHP give zero for $R \to \infty$. The integral I_ϵ around the small circle of radius ϵ, centered on ia, is of order $2\pi\epsilon\, e^{-ka}/\sqrt{\epsilon}$, and this tends to zero for $\epsilon \to 0$.

We note that $I_{B1} = I_{B2}$: the integrand has opposite signs for I_{B1} and I_{B2}, but the two straight-line paths are followed in opposite directions, which reverses the sign a second time.

Thus, in the limit $R \to \infty$ and $\epsilon \to 0$, we have $I = I_{B1} + I_{B2} = 2I_{B2}$.

So far, all we have done is to replace an integral along the real axis by an integral along a portion of the imaginary axis. Fortunately, the latter integral is tractable. We can parametrize the path by $z(t) = i(a+t)$, where t runs from 0 to ∞:

$$I = 2\int_{ia}^{i\infty} \frac{e^{ikz}}{\sqrt{z-ia}}\,\mathrm{d}z = 2\int_0^\infty \frac{e^{-k(a+t)}}{\sqrt{(it)}}\,i\mathrm{d}t = 2 \times \frac{1+i}{\sqrt{2}} \times e^{-ka} \int_0^\infty \frac{e^{-kt}}{\sqrt{t}}\,\mathrm{d}t.$$

The substitution $u^2 = kt$ turns the real integral at the end of the line into a standard Gaussian integral, $\int_0^\infty e^{-u^2}\,\mathrm{d}u = \frac{1}{2}\sqrt{\pi}$. The final result for I is therefore $(1+i)e^{-ka}\sqrt{2\pi/k}$.

41. $I = \displaystyle\int_0^\infty \frac{\sqrt{x}}{(x+1)^2}\,\mathrm{d}x$: *See figure (C) on the first page of these solutions.* We note that the integrand has a branch point at $z = 0$. We take the branch cut along the positive real axis, and the contour cannot cross it. Thus at all points $0 < \theta < 2\pi$. For this integral, we use the contour illustrated in figure (C). I_1 and I_3 are infinitesimally displaced from the real axis, above and

below. However as the integrand has no actual discontinuities there (the choice of the branch cut position is arbitrary) the integrals will be as close as we like to the integral along the real axis, and hence can be used to find I as detailed below.

There is one pole within the contour, a double one at $z=-1$, and the residue there is

$$b_1^{z=-1} = \left.\frac{\mathrm{d}\sqrt{z}}{\mathrm{d}z}\right|_{z=-1} = \left.\frac{1}{2\sqrt{z}}\right|_{z=-1} = -\frac{i}{2}.$$

(There is no sign ambiguity here: because of the branch cut, the position of the pole is $-1 = e^{i\pi}$, not eg $e^{3i\pi}$.)

As the integrand falls off as $1/R^{3/2}$, the integral I_2 tends to zero as $R\to\infty$. Also on the small circle in the vicinity of the origin, if we write $z = \epsilon e^{i\theta}$, we have

$$\lim_{\epsilon\to 0} I_4 = \lim_{\epsilon\to 0}\int_0^{2\pi} \frac{\epsilon^{3/2}e^{3i\theta/2}}{(1+\epsilon e^{i\theta})^2}\mathrm{d}\theta = 0$$

The integral I_3 differs from $-I_1$ because on I_1, $z=x$, but on I_3, $z = xe^{2i\pi}$. Thus $\sqrt{z}$ will differ on the two. Hence $I_3 = -e^{i\pi}I_1 = I_1$. Then

$$I_1 + I_3 = 2I_1 = 2\pi i b_i^{z=-1} - I_2 - I_4 \qquad \text{and} \qquad I = \lim_{R\to\infty,\epsilon\to 0} I_1 = \pi i b_i^{z=-1} = \frac{\pi}{2}$$

42. **Note:** In this question on summation of series via a contour integral, a circular contour has been used in the solutions, rather than the square contour that was used in lectures: this makes the argument very slightly more complicated than in the lectures. *The results, of course, do not depend on the shape of the contour.*

a) $f(z) = \dfrac{\cot z}{z^4} = \dfrac{\cos z}{z^4 \sin z}$ has poles at $z = n\pi$. In lectures we saw that the poles of $1/\sin z$ are simple poles with residues $(-1)^n$; here the residues (for $n\neq 0$) are $(-1)^n(\cos z/z^4)_{z=n\pi} = 1/(\pi n)^4$.

The pole at $z=0$ is 5th order. The easiest way to find the residue is by using the expansions of $\cos z$ and $\sin z$ about zero to obtain

$$\begin{aligned}\frac{\cos z}{z^4\sin z} &= \frac{1}{z^5}\frac{1-z^2/2!+z^4/4!-\dots}{1-z^2/3!+z^4/5!-\dots}\\ &= \frac{1}{z^5}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)\left(1+\frac{z^2}{3!}-\frac{z^4}{5!}+\left(\frac{z^2}{3!}\right)^2+\dots\right)\\ &= \frac{1}{z^5}\left(1-\frac{z^2}{3}-\frac{z^4}{45}+\dots\right)\end{aligned}$$

so the residue at $z=0$ is $-1/45$.

Our contour is a large circle of radius $R = (N+\frac{1}{2})\pi$, so that it crosses the x-axis half way between the poles. We need to show that $\cot z$ doesn't increase with R on this contour; if it does not, the factor of $1/z^4$ will ensure that the integrand falls off fast enough for the integral to tend to zero as $R\to\infty$. Now, we can show that $\tanh|y| \le |\cot z| \le \coth|y|$, so $|\cot z|$ will quickly approach 1 as $|y|$ increases; e.g. for $|y|=3$, the two bounds on $|\cot z|$ are already 0.995 and 1.005. On the other hand, close to where the circle crosses the real axis, $x = (N+\frac{1}{2})\pi + \mathcal{O}(y^2/R)$, so that $\cot z = -i\tanh y + \mathcal{O}(y^2/R)$, whose modulus is certainly bounded. So everywhere on the circle, $|\cot z|$ remains bounded, as required. Thus, as $N\to\infty$, the whole contour integral tends to zero, which gives us

$$\lim_{N\to\infty}\left\{\sum_{n=-N}^{-1}\frac{1}{(\pi n)^4} + \sum_{n=1}^{N}\frac{1}{(\pi n)^4} - \frac{1}{45}\right\} = 0 \quad\Rightarrow\quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

b) $f(z) = \dfrac{1}{z^5 \cos z}$ has poles at $z = 0$ and $z = (n + \frac{1}{2})\pi$. From (31 b) the residues of $1/\cos z$ are $(-1)^{n+1}$ so here the residues are $(-1)^{n+1}/\left((n + \frac{1}{2})\pi\right)^5$.

The pole at $z = 0$ is 5th order. The easiest way to find the residue is by using the expansions of $\cos z$ about zero to obtain

$$\begin{aligned}\frac{1}{z^5 \cos z} &= \frac{1}{z^5} \frac{1}{1 - z^2/2! + z^4/4! - \ldots} \\ &= \frac{1}{z^5}\left(1 + \frac{z^2}{2!} + \frac{z^4}{(2!)^2} - \frac{z^4}{4!} + \ldots\right) = \frac{1}{z^5}\left(1 + \ldots + \frac{5z^4}{24} + \ldots\right)\end{aligned}$$

so the residue at $z = 0$ is $5/24$.

Our contour is a large circle of radius $R = N\pi$, so that it crosses the x-axis half way between the poles. We need to show that $|1/\cos z|$ doesn't increase with R on this contour; if it does not, the factor of $1/z^5$ will ensure that the integrand falls off fast enough for the integral to tend to zero as $R \to \infty$. Now we can show that $1/\cosh|y| \le |1/\cos z| \le 1/\sinh|y|$, so $|\sec z|$ decreases rapidly with increasing $|y|$. On the other hand, close to where the circle crosses the real axis, $x = N\pi + \mathcal{O}(y^2/R)$ and $1/\cos z = (-1)^N/\cosh y + \mathcal{O}(y^2/R)$, whose modulus tends to 1 as $y \to 0$. So everywhere on the circle $|1/\cos z|$ remains bounded, as required. Thus, as $N \to \infty$, the whole contour integral tends to zero, giving us

$$\lim_{N\to\infty}\left\{\sum_{n=-N}^{N} \frac{(-1)^{n+1}2^5}{([2n+1]\pi)^5} + \frac{5}{24}\right\} = 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}.$$

43. We have $w = 1/z$ and $g(w) = g(1/z) \equiv f(z)$. Also, $\mathrm{d}z = -\mathrm{d}w/w^2$ and C' is the curve on the w plane corresponding to the curve C in the z plane. Note that if we traverse C in the conventional, anticlockwise direction, we traverse C' in the opposite direction (as θ increases, $\mathrm{Arg}(w) = -\theta$ becomes more negative.) Hence if we write $\oint_{C'}$ indicating anticlockwise integration, we need an extra minus sign, so

$$\oint_C f(z)\mathrm{d}z = \oint_{C'} \frac{g(w)}{w^2}\mathrm{d}w.$$

The required result that the sum of the residues of $f(z)$ within C must equal the sum of the residues of $g(w)/w^2$ within C' follows immediately from the residue theorem.

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1},$$

so $\oint_C f(z)\mathrm{d}z$ is 0, $-2\pi i$ or 0, for $R = 1/2$, $3/2$ and $5/2$.

$$g(w) = \frac{w^2}{2w^2 - 3w + 1} \quad \Rightarrow \quad \frac{g(w)}{w^2} = \frac{2}{w-1} - \frac{2}{w - \frac{1}{2}},$$

so $\oint_C f(z)\mathrm{d}z$ is 0, $-2\pi i$ or 0, for $R' = 2$, $2/3$ and $2/5$.

Recalling that the singularities of $g(w)$ at $w = 0$ can be interpreted as those of $f(z)$ at $z = \infty$, we see that the residue of $f(z)$ at infinity is just that of $-g(w)/w^2$ at 0. Including poles at infinity allows us to evaluate a contour integral via the residues of all the poles *outside* the contour, rather than those *inside*; this can be useful in cases where it reduces the amount of work to be done in evaluating residues.