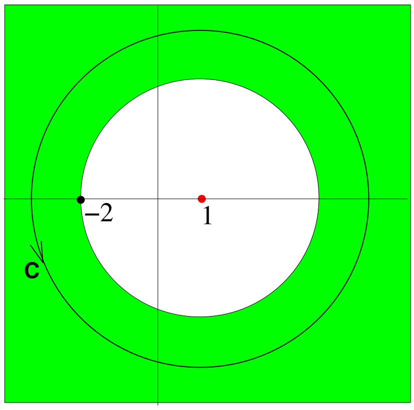
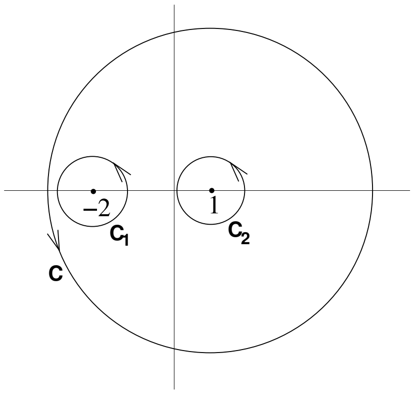
PHYS20672 Complex Variables and Vector Spaces:
Solutions 4

29. The first figure below shows the set-up: the central (red) point is the expansion point $z = 1$, and the shaded (green) area is the region in which the expansion is to be valid. (The outer radius is infinite.) C is a contour lying within the shaded region. The second figure defines the contours C_1 and C_2; their radii are unimportant so long as they are less than 3.





According to Laurent's theorem we can write $f(z)$ as follows:

$$f(z) = \sum_{n=1} \frac{b_n}{(z-a)^n} + \sum_{n=0} a_n (z-a)^n$$

$$\text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z \qquad \text{and} \qquad b_n = \frac{1}{2\pi i}\oint_C f(z)(z-a)^{n-1}\mathrm{d}z$$

First we calculate a_n:

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_C \frac{\mathrm{d}z}{(z+2)(z-1)^{n+1}} \\ &= \frac{1}{2\pi i}\oint_{C_1} \frac{\mathrm{d}z}{(z+2)(z-1)^{n+1}} + \frac{1}{2\pi i}\oint_{C_2} \frac{\mathrm{d}z}{(z+2)(z-1)^{n+1}} \\ &= \left.\frac{1}{(z-1)^{n+1}}\right|_{z=-2} + \left.\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{1}{(z+2)}\right|_{z=1} = \frac{1}{(-3)^{n+1}} + \frac{(-1)^n}{3^{n+1}} = 0 \end{aligned}$$

In the last line we use the following result: for $f(z) = \dfrac{1}{z-z_0}$, $f^{(n)}(z) = \dfrac{(-1)^n n!}{(z-z_0)^{n+1}}$.

So all the a_n vanish—there are no positive powers of $(z-1)$ in the series.

Next we calculate b_n

$$b_n = \frac{1}{2\pi i}\oint_C \frac{(z-1)^{n-1}}{z+2} = (z-1)^{n-1}|_{z=-2} = (-3)^{n-1}$$

So finally

$$\frac{1}{z+2} = \sum_{n=1} \frac{(-3)^{n-1}}{(z-1)^n} = \frac{1}{z-1} - \frac{3}{(z-1)^2} + \frac{9}{(z-1)^3} - \ldots \qquad \text{for } |z-1| > 3.$$

Note that we can get the same result from geometric series with much less effort, as we will see in the next-but-one question.

30. The nearest singularity is at $z = \frac{\pi}{2}$, so the radius of convergence is $z = \frac{\pi}{4}$.
Using $f(z) = \tan z = \sin z / \cos z$ we have $f'(z) = 1 + \tan^2 z$, $f''(z) = 2\tan z + 2\tan^3 z$, $f^{(3)}(z) = 2 + 8\tan^2 z + 6\tan^4 z$; so $f(\frac{\pi}{4}) = 1$, $f'(\frac{\pi}{4}) = 2$, $f''(\frac{\pi}{4}) = 4$ and $f^{(3)}(\frac{\pi}{4}) = 16$ giving

$$\tan z = 1 + 2(z - \tfrac{\pi}{4}) + 2(z - \tfrac{\pi}{4})^2 + \tfrac{8}{3}(z - \tfrac{\pi}{4})^3 + \dots$$

31. From the result given in the last question but one, we have $f^{(n)}(0) = -\dfrac{n!}{z_0^{n+1}}$, and so the Taylor series of $1/(z - z_0)$ is

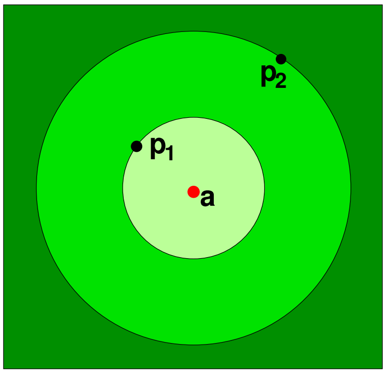
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n = -\sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}} = -\frac{1}{z_0} - \frac{z}{z_0^2} - \frac{z^2}{z_0^3} - \dots$$

Note that we could obtain the same result by writing $f(z) = -(1/z_0)(1 - z/z_0)^{-1}$ and using the result for a geometric series. The second form makes it clear that the radius of convergence is $|z_0|$.

For $|z| > |z_0|$, to obtain the Laurent series we write instead

$$f(z) = \frac{1}{z}\left(1 - \frac{z_0}{z}\right)^{-1} = \sum_{n=0}^{\infty} \frac{z_0^n}{z^{n+1}} = \frac{1}{z} + \frac{z_0}{z^2} + \frac{z_0^2}{z^3} + \dots$$

32.



In the general case of two poles, for a given expansion point there will be three regions with a different series in each, as shown in the figure (where a is the expansion point and p_i are the positions of the poles). The expansions in parts (a), (b) and (c) below are to hold in the light, medium and dark shaded regions respectively; the same is true of parts (e), (f) and (g) but with a different expansion point. The exception is where the expansion point is one of the poles: in that case there is no purely Taylor expansion and so there are only two series (another way of putting it is that radius of the inner region is zero). Part (d) below is an example of expansion about a pole.

We use partial fractions to write

$$\frac{z+1}{(z-2)(z-3)} = \frac{4}{(z-3)} - \frac{3}{(z-2)}.$$

Then we choose either the Laurent or Taylor series for each term independently, depending on whether or not its pole lies between the expansion point and the region we want the expansion to hold in.

a) about $z = 0$, for the region $|z| < 2$: Both poles lie beyond this region, so we use the Taylor series in each case:

$$f(z) = -4\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} + 3\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} z^n \left(\frac{3}{2^{n+1}} - \frac{4}{3^{n+1}}\right) = \frac{1}{6} + \frac{11}{36}z + \frac{49}{216}z^2 + \dots$$

b) about $z = 0$, for the region $2 < |z| < 3$: Taylor series for the first term again, but Laurent series for the second because the pole at $z = 2$ lies between the expansion point and the region of validity of the expansion:

$$f(z) = -4\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} - 3\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = -\frac{4}{3} - \frac{4}{9}z - \frac{4}{27}z^2 + \dots - \frac{3}{z} - \frac{6}{z^2} - \frac{12}{z^3} + \dots$$

c) about $z = 0$, for the region $3 < |z|$: Laurent series for both:

$$f(z) = 4\sum_{n=0}^{\infty} \frac{3^n}{z^{n+1}} - 3\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} z^{-(n+1)}\left(4 \times 3^n - 3 \times 2^n\right) = \frac{1}{z} + \frac{6}{z^2} + \frac{24}{z^3} + \ldots$$

d) about $z = 2$, for the region $1 < |z - 2|$: write $w = z - 2$, so $f(w) = 4/(w-1) - 3/w$. For $|w| > 1$ we need a Laurent series for $4/(w-1)$, giving

$$f(z) = 4\sum_{n=0}^{\infty} \frac{1}{w^{n+1}} - \frac{3}{w} = \frac{1}{z-2} + \frac{4}{(z-2)^2} + \frac{4}{(z-2)^3} + \ldots$$

e) about $z = 1$: write $w = z - 1$, so $f(w) = 4/(w-2) - 3/(w-1)$. For the region $|z-1| < 1$: Taylor series in each case,

$$f(z) = -4\sum_{n=0}^{\infty} \frac{w^n}{2^{n+1}} + 3\sum_{n=0}^{\infty} w^n = 1 + 2(z-1) + \frac{5}{2}(z-1)^2 + \ldots$$

f) about $z = 1$, for the region $1 < |z-1| < 2$: Taylor series for the first term, but Laurent series for the second:

$$f(z) = -4\sum_{n=0}^{\infty} \frac{w^n}{2^{n+1}} - 3\sum_{n=0}^{\infty} \frac{1}{w^{n+1}} = -2 - (z-1) - \frac{1}{2}(z-1)^2 + \ldots - \frac{3}{z-1} - \frac{3}{(z-1)^2} - \ldots$$

g) about $z = 1$, for the region $2 < |z-1|$: Laurent series for both:

$$f(z) = 4\sum_{n=0}^{\infty} \frac{2^n}{w^{n+1}} - 3\sum_{n=0}^{\infty} \frac{1}{w^{n+1}} = \frac{1}{z-1} + \frac{5}{z-1^2} + \frac{13}{z-1^3} + \ldots$$

In the following, when we say we work to nth order we include terms up to z^n and z^{-n}. The exact results are $f(1) = 1$, $f(2.5) = -14$ and $f(6) = 7/12 = 0.5833$. To get 1% accuracy we need to work to 8th order in series (a) for $z = 1$ (result 0.994). If you were to use series (b) or (c) at $z = 1$ the corresponding results would be would be -767 and 12355, respectively. For $z = 2.5$, the 8th order result in series (b) is -11.44 and we need to go to 25th order to get -13.91. (The 8th order results in (a) and (c), in contrast, are 32.25 and 21.4, while the 25th order results are 1971 and 749.) Finally, for $z = 6$, the 8th order result in series (c) is 0.578. (For (a) and (b) we get 14080 and -682). This should illustrate the fact that the correct series is convergent—albeit slowly for (b)—but the wrong series is not!

33. a) Let $w = 1/z$. Since the Taylor series for $\sin w$ converges for all finite w, it will also converge for all $z \neq 0$. So

$$\sin w = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \ldots \qquad \Rightarrow \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots$$

b) Using $\sin^2 z = \frac{1}{2}(1 - \cos 2z)$ and $\cos w = 1 - w^2/2! + w^4/4! - \ldots$ gives

$$z^{-3}\sin^2 z = \frac{1}{2z^3}\left(\frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} - \ldots\right) = \frac{1}{z} - \frac{1}{3}z + \frac{2}{45}z^3 + \ldots$$

c) As in (a), we can use the Taylor series for e^w, with $w = 1/z$, to get

$$z^3 e^{1/z} = z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{4!z} + \frac{1}{5!z^2} \ldots$$

d)

$$\frac{\cos z - 1}{z^2} = -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!}$$

Singularities and residues: (a) essential singularity, $b_1 = 1$; (b) simple pole, $b_1 = 1$; (c) essential singularity, $b_1 = 1/24$. In case (d), we say that the function $f(z)$ has a 'removable' singularity: although the ratio $0/0$ doesn't exist, we can define $f(0)$ to be the limit $\lim_{z\to 0} f(z)$; our series expansion shows that the resulting function is analytic at $z = 0$, with $b_1 = 0$.

34. By inspection, $f(z)$ has two singularities: a simple pole at $z = 0$ and a pole of order 3 at $z = 1$. We can use Cauchy's integral formulae directly to find the residues; equivalently we write as follows:

At $z = 0$ the residue is $\displaystyle \lim_{z\to 0}\,(zf(z)) = \left.\frac{z^2+1}{(z-1)^3}\right|_{z=0} = -1$

At $z = 1$ the residue is $\displaystyle \lim_{z\to 1}\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left((z-1)^3 f(z)\right) = \frac{1}{2}\left(\frac{2}{z^3}\right)_{z=1} = 1$

35. a) At first glance $\dfrac{z^2+z-2}{(z-1)^2}$ appears to have a double pole at $z = 1$. We can use

$$b_1 = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}(z^2+z-2) = \left.2z+1\right|_{z=1} = 3$$

Had we spotted that the numerator factorises: $z^2 + z - 2 = (z + 2)(z - 1)$, we would have seen that it is actually a single pole and the value of the residue would have been obvious by inspection. This demonstrates that we won't go wrong if we overestimate the order of the pole.

b) The poles of $1/\cos z$ are at the zeros of $\cos z$, namely $z = (n+\frac{1}{2})\pi$ for integer n. Making the substitution $z = w + (n+\frac{1}{2})\pi$ we have $\cos z = \cos(w + (n+\frac{1}{2})\pi) = (-1)^{n+1}\sin w$. The residue at $w = 0$ of $1/\sin w$ is $\lim_{w\to 0}(w/\sin w) = 1$. Hence the residues of $1/\cos z$ at $z = (n+\frac{1}{2})\pi$ are $(-1)^{n+1}$.

c) The poles of $z/\sin^2 z$ are at the zeros of $\sin z$. A sketch of $\sin^2 z/z$ shows that it has a simple zero at $z = 0$ and double zeros at $z = n\pi$ for $n \neq 0$. The residue at $z = 0$ is $\lim_{z\to 0}\big(zf(z)\big) = \lim_{z\to 0}(z^2/\sin^2 z) = 1$.
At $z = n\pi$, we write $z = w + n\pi$ and $\sin^2 z = \sin^2 w$. Hence we have

$$\frac{z}{\sin^2 z} = \frac{w}{\sin^2 w} + \frac{n\pi}{\sin^2 w}$$

The second term contains only even powers of w and so has no residue, but the first term has the form we've already considered and so has residue 1 at $w = 0$. Hence the residues of $z/\sin^2 z$ at $z = n\pi$ are all 1.

d) $(\sin z - \cos z)^{-1}$ has poles at $z = (n+\frac{1}{4})\pi$. Setting $z = w + (n+\frac{1}{4})\pi$ we write $\sin z - \cos z = \sqrt{2}\sin(z - \pi/4) = \sqrt{2}\sin(w + n\pi) = (-1)^n\sqrt{2}\sin w$. (Direct substitution without the trick in the first step gives the same result after more algebra.) Hence the residues of are $(-1)^n/\sqrt{2}$.

e) The zeros of $\sinh z$ are on the imaginary axis, since $\sinh iy = i\sin y$. So $1/\sinh z$ has simple poles at $z = in\pi$. Since $\sinh(w + in\pi) = (-1)^n\sinh w$ and $\lim_{w\to 0}(w/\sinh w) = 1$, the residues of $1/\sinh z$ at $z = in\pi$ are $(-1)^n$. Note all of these functions also have essential singularities at $z = \infty$. The sign of this is that $\lim_{z\to\infty} f(z)$ does not exist; in fact its behaviour for large $|z|$ depends on direction, i.e. $\arg z$.