PHYS20672 Complex Variables and Vector Spaces: Solutions 3

21. For C_1, $y = 0$ and $\mathrm{d}y = 0$ while x runs from 1 to 0, then $x = 0$ and $\mathrm{d}x = 0$ while y runs from 0 to 1. For C_2, $y = 1 - x$ and $\mathrm{d}y = -\mathrm{d}x$. We want the integral $\int u\,\mathrm{d}x - v\,\mathrm{d}y + i\int v\,\mathrm{d}x + u\,\mathrm{d}y$ along each path.
    a) $u = x$ and $v = 0$:
$$\int_{C_1} x\,\mathrm{d}x + i\int_{C_1} x\,\mathrm{d}y = \int_1^0 x\,\mathrm{d}x + i\int_0^1 0\,\mathrm{d}y = -\tfrac{1}{2}$$
$$\int_{C_2} x\,\mathrm{d}x + i\int_{C_2} x\,\mathrm{d}y = \int_1^0 (x - ix)\,\mathrm{d}x = -\tfrac{1}{2}(1-i)$$
    b) $u = x$ and $v = y$:
$$\int_{C_1} x\,\mathrm{d}x - y\,\mathrm{d}y + i\int_{C_1} x\,\mathrm{d}y + y\,\mathrm{d}x = \int_1^0 x\,\mathrm{d}x - \int_0^1 y\,\mathrm{d}y = -1$$
$$\int_{C_2} x\,\mathrm{d}x - y\,\mathrm{d}y + i\int_{C_2} x\,\mathrm{d}y + y\,\mathrm{d}x = \int_1^0 1 + i(1-2x)\,\mathrm{d}x = -1$$
    We note that (b) is path-independent and agrees with $[z^2/2]_1^i$. This is as expected because z is an analytic funtion (unlike $\mathrm{Re}\, z$).

22. For $f(z) = |z|$, $u = \sqrt{x^2+y^2}$ and $v = 0$. On C_1, $y = 0$ and $\mathrm{d}y = 0$; thus only $\int u\mathrm{d}x$ is non zero. x runs from 1 to -1.
$$\int_{C_1} \sqrt{x^2+y^2}\,\mathrm{d}x = \int_1^{-1} |x|\,\mathrm{d}x = -1$$
    On C_2, $z = e^{i\theta}$, $|z| = 1$ and $\mathrm{d}z = iz\,\mathrm{d}\theta$:
$$\int_{C_2} 1\,\mathrm{d}z = i\int_0^{\pi} e^{i\theta}\,\mathrm{d}\theta = [e^{i\theta}]_0^{\pi} = -2$$
    Again, we would not have expected these to be the same.

23. Along the unit circle centred on a, $z = a + e^{i\theta}$ and $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta$
$$\oint_C \frac{1}{z-a}\,\mathrm{d}z = i\int_0^{2\pi} \frac{1}{e^{i\theta}}\,e^{i\theta}\,\mathrm{d}\theta = i[\theta]_0^{2\pi} = 2\pi i$$
$$\oint_C \frac{1}{(z-a)^n}\,\mathrm{d}z = i\int_0^{2\pi} \frac{1}{e^{in\theta}}\,e^{i\theta}\,\mathrm{d}\theta = \left[\frac{e^{i(1-n)\theta}}{1-n}\right]_0^{2\pi} = 0 \qquad \text{for integer } n > 1$$
    By Cauchy's theorem, these results will hold for any contour enclosing the point $z = a$, not just the unit circle. For future reference we note also that $\oint (z-a)^n \mathrm{d}z = 0$ for $n \geq 0$, since the integrand is everywhere analytic.

In the following problems, we first use partial fractions to express the integrand as a sum of terms which resemble those above. Since $\oint \big(af(z) + bg(z)\big)\,\mathrm{d}z = a\oint f(z)\,\mathrm{d}z + b\oint g(z)\,\mathrm{d}z$, we can sum the weighted contributions from each part. These contributions will then depend solely on whether or not the pole of the integrand is within the contour.

(a)

$$f(z) = \frac{1}{z-i}$$

The only pole is at $z = i$.
(i) $R = 1/2$: the pole is outside the contour, so the function is analytic inside the contour and the integral is 0.
(ii) $R = 2$: the pole is inside the contour and the integral is $2\pi i$.

(b)

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $R = 1/2$: both poles are outside the contour, so the integral is 0.
(ii) $R = 3/2$: only the pole at $z = 1$ is inside, so the integral is $-2\pi i$.
(iii) $R = 5/2$: both poles are inside the contour, so the integral is $2\pi i(-1+1) = 0$.

(c)

$$f(z) = \frac{z+1}{z^2 - 3z + 2} = \frac{3}{z-2} - \frac{2}{z-1}$$

Here the contributions from the poles at $z = 1$ and $z = 2$ are multiplied by -2 and 3 respectively, so the answers are (i) 0, (ii)$-4\pi i$, (iii) $(3-2) \times 2\pi i = 2\pi i$.

(d)

$$f(z) = \frac{z^2 + z + 1}{z^3 - z^2} = -\frac{2}{z} - \frac{1}{z^2} + \frac{3}{z-1}$$

Here the second term $1/z^2$ does not contribute to the integral, where ever the contour lies. So the answers are (i) $-4\pi i$ and (ii) $2\pi i$.

24. In each case, we compare the integrand to $f(z)/(z-a)^n$, and use the Cauchy formula to give the answer $2\pi i f^{(n-1)}(a)/(n-1)!$ — or just $2\pi i f(a)$ for a simple pole ($n = 1$). It is important to realise that the analytic part "$f(z)$" may not all be in the numerator.
a) Here, there is a simple pole at $z = 0$ and it is inside the contour C_1. So "a" $= 0$, "$f(z)$" $= e^{3z}$ and the integral is just $2\pi i\, e^0 = 2\pi i$.
b) Here there is a double pole inside the contour C_1 at $z = 0$ (so "a" $= 0$). Also "$f(z)$" $= \cos^2 2z$, with first derivative $= -4\cos 2z \sin 2z$ which vanishes at $z = 0$. So the integral is zero.
c) Again, the integral is zero. (The sum of (b) and (c) is $\oint_{C_1}(1/z^2)\mathrm{d}z = 0$.)
d) The simple pole at $z = 2i$ is inside the contour C_2. So "a" $= 2i$, "$f(z)$" $= z^2$ and the integral is $2\pi i(2i)^2 = -8\pi i$.
e) As the denominator can be written $(z-2i)(z+2i)$, there are simple poles at $z = \pm 2i$, but only the one at $z = 2i$ is inside the contour C_2. So "a" $= 2i$. We identify "$f(z)$" with $z^2/(z+2i)$, which is analytic within C_2, giving

$$\oint_{C_2} \frac{z^2}{z^2+4}\mathrm{d}z = \oint_{C_2} \frac{z^2}{(z+2i)(z-2i)}\mathrm{d}z = 2\pi i \left.\frac{z^2}{z+2i}\right|_{z=2i} = 2\pi i \frac{(2i)^2}{4i} = -2\pi.$$

25. Recalling that $|z_1 + z_2| \geq |z_1| - |z_2|$ (for $|z_1| \geq |z_2|$), we have $|z^2+1| \geq |z|^2 - |1| = R^2 - 1$. Hence $|z^2+1|^{-1} \leq (R^2-1)^{-1}$. Now by the estimation lemma, for the circular path $|z| = R$,

$$\left|\oint \frac{1}{z^2+1}\mathrm{d}z\right| \leq 2\pi R \frac{1}{R^2-1}$$

Hence the limit as $R \to \infty$ is zero. Furthermore if the magnitude of the integral is bounded from above by zero, the integral itself must tend to zero.
The integrand is proportional to $1/(z-i) - 1/(z+i)$, so by Cauchy's formula the integral along any circular path $|z| = R$, with $R > 1$, will pick up cancelling contributions from the two poles and will be zero.

26. In the following, the contour C is the unit circle centred on $z = 0$. In parts (a)-(c) (and in some subsequent questions) we will need the binomial theorem: for positive integer n,

$$(a+b)^n = a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \ldots + nab^{n-1} + b^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m$$

where $\displaystyle \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$.

a)

$$\int_0^{2\pi} \cos^4\theta \,\mathrm{d}\theta = \frac{1}{16i} \oint_C \left(z + \frac{1}{z}\right)^4 \frac{\mathrm{d}z}{z} = \frac{1}{16i} \oint_C \left(z^3 + 4z + \frac{6}{z} + \frac{4}{z^3} + \frac{1}{z^5}\right) \mathrm{d}z = 2\pi i \frac{6}{16i} = \frac{3\pi}{4}$$

Here we have used the fact that $\oint z^n \mathrm{d}z = 0$ unless $n = -1$ (see qu. 22). As a useful check, we see that a positive definite integral has given a real, positive result!

b)

$$\int_0^{2\pi} \sin^6\theta \,\mathrm{d}\theta = \frac{1}{-64i} \oint_C \left(z - \frac{1}{z}\right)^6 \frac{\mathrm{d}z}{z} = -\frac{1}{64i} \oint_C \left(z^5 + \ldots - \frac{20}{z} + \ldots\right) \mathrm{d}z = 2\pi i \frac{20}{64i} = \frac{5\pi}{8}$$

c)

$$\int_0^{2\pi} \cos^{2n}\theta \,\mathrm{d}\theta = \frac{1}{2^{2n}i} \oint_C \left(z + \frac{1}{z}\right)^{2n} \frac{\mathrm{d}z}{z} = \frac{1}{2^{2n}i} \oint_C \left(z^{2n-1} + \ldots + \binom{2n}{n}\frac{1}{z} + \ldots\right) \mathrm{d}z$$
$$= \frac{2\pi i}{2^{2n}i} \frac{(2n)!}{n!n!} = \frac{2\pi}{2^{2n}} \frac{(2n)!!(2n-1)!!}{n!n!} = \frac{2\pi}{2^n} \frac{(2n-1)!!}{n!} = 2\pi \frac{(2n-1)!!}{(2n)!!}$$

where we have used $(2n)!! = 2^n n!$. (In (b) and (c) we have exploited the fact that the only contribution will come from the term proportional to $1/z$ to avoid to calculating the coefficients of the other terms.)

d)

$$\int_0^{2\pi} \frac{\cos\theta}{4\cos\theta - 5} \mathrm{d}\theta = \frac{1}{2i} \oint_C \frac{z + z^{-1}}{2(z + z^{-1}) - 5} \frac{\mathrm{d}z}{z} = \frac{1}{2i} \oint_C \frac{z^2 + 1}{z(2z^2 - 5z + 2)} \mathrm{d}z$$
$$= \frac{1}{4i} \oint_C \frac{z^2 + 1}{z(z - \frac{1}{2})(z - 2)} \mathrm{d}z$$

Now the integrand has simple poles at $z = 0$, $z = 1/2$ and $z = 2$, but only the first two lie within the contour. By Cauchy's theorem, we can replace the integral round the unit circle with two smaller contours, C_1 circling $z = 0$ and C_2 circling $z = 1/2$.

To evaluate each of these we use Cauchy's integral formula for $\oint f(z)/(z-a)\mathrm{d}z$, all the terms except the relevant pole being part of "$f(z)$" (as in question (23 e) above). This gives

$$\frac{1}{4i} \oint_C \frac{z^2 + 1}{z(z - \frac{1}{2})(z - 2)} \mathrm{d}z = \frac{2\pi i}{4i} \left(\left.\frac{z^2 + 1}{(z - \frac{1}{2})(z - 2)}\right|_{z=0} + \left.\frac{z^2 + 1}{z(z - 2)}\right|_{z=\frac{1}{2}} \right) = -\frac{\pi}{3}$$

e) Here, we use $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + z^{-2})$. We end up with a double pole at $z = 0$ and a simple pole at $z = -1/3$ within the unit circle, and evaluate the contribution to each separately as in the previous part:

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2\theta}{3\cos\theta + 5}\,\mathrm{d}\theta &= \frac{1}{2i} \oint_C \frac{z^2 + z^{-2}}{\frac{3}{2}(z + z^{-1}) + 5}\,\frac{\mathrm{d}z}{z} = \frac{1}{3i} \oint_C \frac{z^4 + 1}{z^2(z^2 + 10z/3 + 1)}\mathrm{d}z \\
&= \frac{1}{3i} \oint_C \frac{z^4 + 1}{z^2(z + \frac{1}{3})(z + 3)}\mathrm{d}z = \frac{2\pi i}{3i} \left(\frac{\mathrm{d}}{\mathrm{d}z} \frac{z^4 + 1}{(z + \frac{1}{3})(z + 3)}\bigg|_{z=0} + \frac{z^4 + 1}{z^2(z + 3)}\bigg|_{z=-\frac{1}{3}} \right) \\
&= \frac{2\pi}{3} \left(-\frac{10}{3} + \frac{41}{12} \right) = \frac{\pi}{18}
\end{aligned}$$

Again, in all the above a real integral has given a real result—a useful check.

27. We *assume* that $f^{(n-1)}(a) = \dfrac{(n-1)!}{2\pi i} \displaystyle\oint \frac{f(z)}{(z-a)^n}\mathrm{d}z$. Then, using the definition of the derivative,

$$\begin{aligned}
f^{(n)}(a) &\equiv \lim_{h\to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{h} \\
&= \lim_{h\to 0} \frac{(n-1)!}{2\pi i h} \left(\oint \frac{f(z)}{(z-a-h)^n}\mathrm{d}z - \oint \frac{f(z)}{(z-a)^n}\mathrm{d}z \right) \\
&= \lim_{h\to 0} \frac{(n-1)!}{2\pi i h} \oint \frac{f(z)\big((z-a)^n - (z-a-h)^n\big)}{(z-a-h)^n (z-a)^n}\mathrm{d}z.
\end{aligned}$$

By using the binomial theorem for $(z-a)^n = ([z-a-h] + h)^n$, this becomes (after a little rearrangement)

$$f^{(n)}(a) = \lim_{h\to 0} \frac{(n-1)!}{2\pi i} \sum_{k=1}^{n} \binom{n}{k} h^{k-1} \oint \frac{f(z)}{(z-a-h)^k (z-a)^n}\mathrm{d}z.$$

The terms with $k > 1$ all vanish as $h \to 0$: each one contains a positive power of h multiplied by an integral whose limit for $h \to 0$ is finite. The only surviving term, with $k = 1$, then gives us

$$f^{(n)}(a) = \frac{(n-1)! \times n}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z.$$

So we've shown that *if* the Cauchy integral formula is true for $n-1$, *then* it is also true for n, and hence for $n+1$, $n+2$, ... But we have shown in lectures that it is true for $n = 0$ (and 1), so we can conclude that it holds for all positive n.

28. The generalized argument theorem states that $\oint [f'(z)/f(z)]\mathrm{d}z = 2\pi i(N - P)$ where N and P are the number of zeros and poles of f respectively within the contour C.
For $f(z) = \dfrac{2z+1}{(z+3)(z-2)}$ we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z + \frac{1}{2}} - \frac{1}{z-2} - \frac{1}{z+3}$$

and so the integral round the contour $|z| = 5/2$ is $2\pi i(1-1) = 0$. But that agrees with the predictions of the argument theorem, given that $f(z)$ has one pole (at $z = 2$) and one zero (at $z = -\frac{1}{2}$) within the contour.