PHYS20672 Complex Variables and Vector Spaces: Solutions 2

10. a) $f(z) = z^3 + z^2$:

$$\begin{aligned} f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 + (z+\Delta z)^2 - z^3 - z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{3z^2\Delta z + 3z(\Delta z)^2 + z(\Delta z)^3 - 2z\Delta z - (\Delta z)^2}{\Delta z} \\ &= 3z^2 + 2z \end{aligned}$$

Here $u(x,y) = x^3 + x^2 - 3xy^2 - y^2$ and $v(x,y) = 3x^2y + 2xy - y^3$. So

$$\frac{\partial u}{\partial x} = 3x^2 + 2x - 3y^2 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -6xy - 2y = -\frac{\partial v}{\partial x}$$

so the Cauchy-Riemann equations are satisfied everywhere.

b) $f(z) = 1/z$ $(z \neq 0)$:

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{1/(z+\Delta z) - 1/z}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \frac{z - (z+\Delta z)}{z(z+\Delta z)} = -\frac{1}{z^2} \end{aligned}$$

Here $u(x,y) = x/(x^2+y^2)$ and $v(x,y) = -y/(x^2+y^2)$. Hence

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

so the Cauchy-Riemann equations are satisfied everywhere except at $z = 0$.

c) $f(z) = |z|^2$:

$$\begin{aligned} f'(z) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{\Delta x + i\Delta y} \end{aligned}$$

We do not need to simplify this further. Since the numerator is real, if $\Delta y = 0$ the limit is real but if $\Delta x = 0$ the limit is imaginary. This path-dependence means that the derivative does not exist.

Here $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Hence the Cauchy-Riemann equations are not satisfied. (Actually they are satisfied at $z = 0$, but not in any neighbourhood of $z = 0$, so the function is not analytic even there.)

11. a) $f(z) = \sin z$ so $u = \sin x \cosh y$ and $v = \cos x \sinh y$. The Cauchy-Riemann equations are satisfied (check!), and hence we can write:

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \cos x \cosh y - i \sin x \sinh y = \cos x \cos(iy) - \sin x \sin(iy) = \cos(x+iy) = \cos z. \end{aligned}$$

b) $f(z) = \ln z$ so $u = \frac{1}{2}\ln(x^2+y^2)$ and $v = \arctan(y/x)$. The Cauchy-Riemann equations are satisfied (remember $\mathrm{d}(\arctan x)/\mathrm{d}x = 1/(1+x^2)$), and hence we can write:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z}.$$

(Note that the point $z = 0$ is not in the domain of the function.) This is much easier using the polar form of the CR equations (see below)!

12. Let $f(z) = w = u(x,y) + iv(x,y)$ and $g(w) = s(u,v) + it(u,v)$. Then the Cauchy-Riemann equations for $g(w)$ are

$$\frac{\partial s}{\partial u} = \frac{\partial t}{\partial v} \quad \text{and} \quad \frac{\partial s}{\partial v} = -\frac{\partial t}{\partial u}$$

We know that $g(z)$ is an analytic function of z; the question is whether $g\big(f(z)\big)$ is also an analytic function of z. Using the chain rule for partial differentiation, we have

$$\begin{aligned}\frac{\partial s}{\partial x} &= \frac{\partial s}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{\partial t}{\partial v}\frac{\partial v}{\partial y} + \left(-\frac{\partial t}{\partial u}\right)\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial t}{\partial y},\end{aligned}$$

where in the second line we used the Cauchy-Riemann equation for $g(w)$ and $f(z)$. So the first Cauchy-Riemann equation for $g\big(f(z)\big)$ is satisfied, and the proof of the other follows the same lines.

13. i) By using the Cauchy-Riemann equations and changing the order of partial differentiation we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

and similarly for v. Hence both satisfy Laplace's equation $\nabla^2\phi(x,y) = 0$.

ii) Using $x = r\cos\theta$ and $y = r\sin\theta$, and using the Cartesian Cauchy-Riemann equations, we have as required

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y} = r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial r} &= \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial \theta} = -r\sin\theta\frac{\partial v}{\partial y} - r\cos\theta\frac{\partial v}{\partial x}\end{aligned}$$

[If you worked in the opposite direction, e.g. by expressing $\partial/\partial x$ in terms of $\partial/\partial r$ and $\partial/\partial\theta$, you might find that you have the *reciprocals* of sin and cos in your results. If so, it's likely that you incorrectly replaced (say) $(\partial r/\partial x)_y$ by $1/(\partial x/\partial r)_\theta$. These quantities are **not** equal, because the variables "held constant" are not the same.]

iii) We have $\partial x/\partial z|_{\overline{z}} = \partial x/\partial\overline{z}|_z = \frac{1}{2}$ and $\partial y/\partial z|_{\overline{z}} = -\partial y/\partial\overline{z}|_z = -\frac{1}{2}i$. So,

$$\begin{aligned}\left.\frac{\partial f}{\partial \overline{z}}\right|_z &= \left.\frac{\partial f}{\partial x}\right|_y\left.\frac{\partial x}{\partial \overline{z}}\right|_z + \left.\frac{\partial f}{\partial y}\right|_x\left.\frac{\partial y}{\partial \overline{z}}\right|_z \\ &= \tfrac{1}{2}\left(\left.\tfrac{\partial u}{\partial x}\right|_y + i\left.\tfrac{\partial v}{\partial x}\right|_y + i\left.\tfrac{\partial u}{\partial y}\right|_x - \left.\tfrac{\partial v}{\partial y}\right|_x\right) = 0\end{aligned}$$

(where we used the Cauchy-Riemann equations at the final stage), and

$$\begin{aligned}\left.\frac{\partial f}{\partial z}\right|_{\overline{z}} &= \left.\frac{\partial f}{\partial x}\right|_y\left.\frac{\partial x}{\partial z}\right|_{\overline{z}} + \left.\frac{\partial f}{\partial y}\right|_x\left.\frac{\partial y}{\partial z}\right|_{\overline{z}} \\ &= \tfrac{1}{2}\left(\left.\tfrac{\partial u}{\partial x}\right|_y + i\left.\tfrac{\partial v}{\partial x}\right|_y - i\left.\tfrac{\partial u}{\partial y}\right|_x + \left.\tfrac{\partial v}{\partial y}\right|_x\right) = \left.\tfrac{\partial u}{\partial x}\right|_y + i\left.\tfrac{\partial v}{\partial x}\right|_y = \tfrac{\mathrm{d}f}{\mathrm{d}z}\end{aligned}$$

Hence the functions such as $\overline{z}$ and $|z|^2 = z\overline{z}$ are not analytic, as we have already shown.

14. First, we note that $u(x,y) = 2xy$ is harmonic, so there must exist a function $f(z)$ of which it is the real part. Using the Cauchy-Riemann equations, we have

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = 2y \Rightarrow v = y^2 + \alpha(x) \\ \text{and} \quad \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = -2x \Rightarrow v = -x^2 + \beta(y) \\ &\Rightarrow v = y^2 - x^2 + c \end{aligned}$$

where c is a real constant (since v is real). Note that in each step, the "constants of integration" α and β, are actually functions of the other variable, and they can only be determined by comparing the results from both integrations.

By inspection we can see that u and v are the real and imaginary parts of $f(z) = -iz^2 + ic$. (If inspection fails, you can also get this by substituting $x = z - iy$ into $u + iv$. Provided you haven't made a mistake, the dependence on y should disappear during simplification, leaving only the required function of z.)

15. $v(x,y) = ye^x \cos y + xe^x \sin y$ so

$$\frac{\partial^2 v}{\partial x^2} = e^x(y\cos y + (2+x)\sin y) = -\frac{\partial^2 v}{\partial y^2}.$$

So v is harmonic, and there must exist a function $f(z)$ of which it is the imaginary part. Using the Cauchy-Riemann equations, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = e^x((1+x)\cos y - y\sin y) \Rightarrow u = e^x(x\cos y - y\sin y) + \beta(y) \\ \text{and} \quad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -e^x(y\cos y + (1+x)\sin y) \Rightarrow u = e^x(x\cos y - y\sin y) + \alpha(x) \\ &\Rightarrow u = e^x(x\cos y - y\sin y) + c. \end{aligned}$$

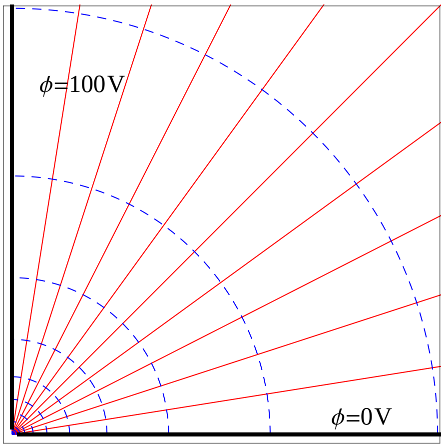
So $w = u + iv = e^x(x+iy)(\cos y + i\sin y) + c = ze^z + c$.

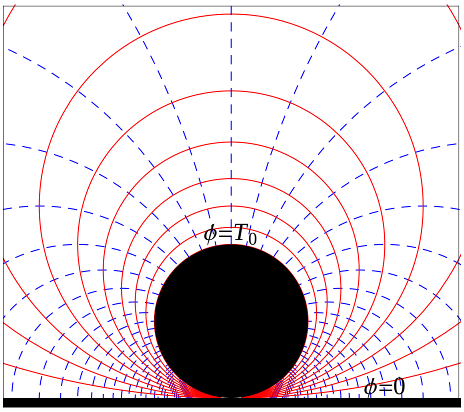
16. Using the CR equations, the Jacobian can be written

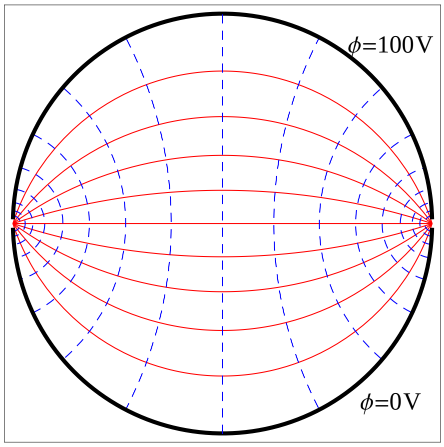
$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

But as $\mathrm{d}f/\mathrm{d}z = \partial u/\partial x + i\partial v/\partial x$, this gives $J = |\mathrm{d}f/\mathrm{d}z|^2$. (Note that the result is expected, because the [linear] scale factor of the transformation $w = f(z)$ is $M = |\mathrm{d}f/\mathrm{d}z|$.)

The next three questions refer to the following plots: equipotentials are red (solid) and field or flow lines are blue (dashed).







17. The geometry is shown in the left-hand plot above. (A tiny gap between the plates at the origin is assumed.) Under the mapping $Z = \ln z$, the positive x-axis maps to the full X-axis (the images of the point $(x, y) = (1, 0)$ is $(X, Y) = (0, 0)$); and the positive y-axis (with $\theta = \pi/2$) maps to a parallel line with $Y = \pi/2$. In the XY-plane we have a parallel plate capacitor with $\phi = (2V_0/\pi)Y$, so the potential in the z-plane is $\phi = (2V_0/\pi)\theta = (2V_0/\pi)\arctan(y/x)$.

18. The geometry is shown in the middle plot above. After re-writing $Z = R^2/z$ as $z = R^2/Z$, the line $Y = 0$ for $X \neq 0$ corresponds to the x-axis, with large values of $|x|$ corresponding to small values of $|X|$, and vice versa. More generally, we have

$$Y = \mathrm{Im}(R^2/z) = -\frac{R^2 y}{x^2 + y^2},$$

which, for $Y \neq 0$, can be rearranged to give $x^2 + y^2 = -R^2 y/Y$. By completing the square for y, this gives

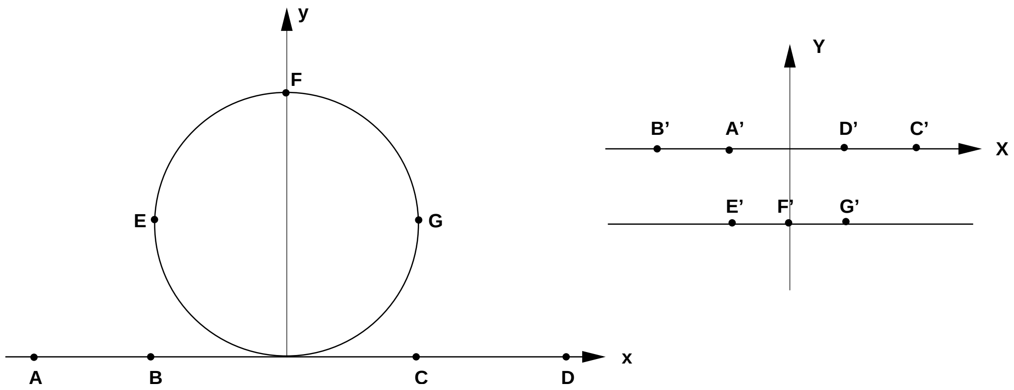
$$x^2 + \left(y + \frac{R^2}{2Y}\right)^2 = \left(\frac{R^2}{2Y}\right)^2,$$

which is the equation of a circle of radius $R^2/(2|Y|)$, centered on $(x_0, y_0) = (0, -R^2/(2|Y|))$. Thus, the line $Y = -R/2$ in the Z plane corresponds to a circle of radius R centered on $(0, R)$ in the z plane.

Accordingly, the geometry in the Z plane is of two parallel plates, and the temperature distribution between them is $\phi = -2T_0 Y/R = 2R^2 y/(x^2 + y^2)$. Equipotentials (red, solid) are circles with radius $\rho > R$ centred on $(0, \rho)$, while heat flow follows circular paths of radius ρ centred on $(\pm\rho, 0)$ (blue, dashed). [The last point can be proved by considering the equation $X = \mathrm{Re}(R^2/z)$.]

To help understand the mapping, the following plot might be useful. Using $X = R^2 x/(x^2 + y^2)$ and $Y = -R^2 y/(x^2 + y^2)$, we can find the images in the XY-plane of the following points as shown in the diagram below:

	A	B	C	D	E	F	G
(x, y)	$(-2R, 0)$	$(-R, 0)$	$(R, 0)$	$(2R, 0)$	$(-R, R)$	$(0, 2R)$	(R, R)
(X, Y)	$(-\frac{1}{2}R, 0)$	$(-R, 0)$	$(R, 0)$	$(\frac{1}{2}R, 0)$	$(-\frac{1}{2}R, -\frac{1}{2}R)$	$(0, -\frac{1}{2}R)$	$(\frac{1}{2}R, -\frac{1}{2}R)$



19. The mapping is $Z = \ln\left(\dfrac{R+z}{R-z}\right)$, with $Y = \arg\left(\dfrac{R+z}{R-z}\right)$. Noting that $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$, and $\arg(R \pm z) = \pm\arctan\left(\dfrac{y}{R \pm x}\right)$, we have

$$Y = \arctan\left(\frac{y}{R+x}\right) + \arctan\left(\frac{y}{R-x}\right).$$

Then, by using

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \quad\Rightarrow\quad \arctan a + \arctan b = \arctan\left(\frac{a+b}{1-ab}\right),$$

we obtain $Y = \arctan\left(\dfrac{2yR}{R^2 - x^2 - y^2}\right)$.

Lines of constant $Y = A$, with $-\pi/2 < A < \pi/2$, have the equation $R^2 - x^2 - y^2 = 2ByR$, with $B = 1/\tan A$, $-\infty < B < \infty$. This can be rearranged to give $x^2 + (y + BR)^2 = (1 + B^2)R^2$, which is an arc of a circle with radius $R\,|\mathrm{cosec}\,A| > R$ and centre $(x, y) = (0, -\cot A)$. For positive A, this arc is in the upper half plane. Note that all the arcs converge on the points $(x, y) = (\pm R, 0)$, which are singular points of the mapping. The line $Y = 0$ is the x-axis ($y = 0$). These lines (for equal increments of Y) are shown in red (solid) in the right-most plot above.

As $|z| \to R$, the argument of the arctan tends to $\pm\infty$ depending on whether $y > 0$ or $y < 0$, and so $Y \to \pm\pi/2$, with the plus sign being for the upper semicircle and the minus sign for the lower semicircle. The image of the point $(x, y) = (R\cos\theta, R\sin\theta)$, is $(X, Y) = \left(\ln\left(\cot\frac{\theta}{2}\right), \pm\frac{\pi}{2}\right)$, and X ranges from ∞ to $-\infty$.

From the results above, we see that the given mapping maps the cylindrical capacitor to a parallel plate capacitor in the XY plane, with plates at $Y = \pm\pi/2$, the upper plate being at 100 V and the lower at 0 V. The potential in such a capacitor is $\phi(X, Y) = (50 + 100Y/\pi)$ V, which leads to the expression in the question for the potential in the original geometry.

The equipotentials (red, solid) and field lines (blue, dashed) are shown on the diagram. The latter are actually lines of constant $\mathrm{Ln}\left(\dfrac{(R+x)^2 + y^2}{(R-x)^2 + y^2}\right)$; they are also arcs of circles, which cross the equipotentials at right angles.

20. The electric field has components $\mathbf{E} = (-\partial u/\partial x, -\partial u/\partial y) = (-\partial u/\partial x, \partial v/\partial x)$, where the y component has been re-written using one of the CR relations. So $E^2 = (\partial u/\partial x)^2 + (\partial v/\partial x)^2 = |\partial f/\partial x|^2$. But $\partial f/\partial x = \mathrm{d}f/\mathrm{d}z$, which gives the final result stated in the question.