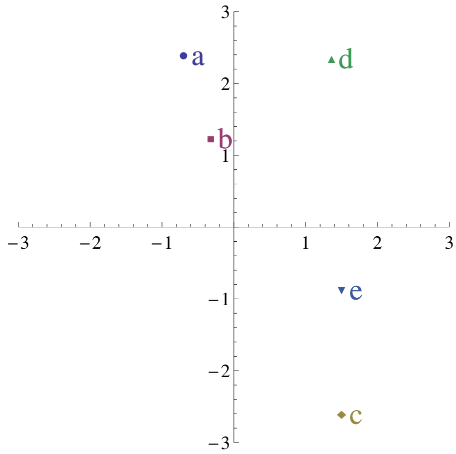
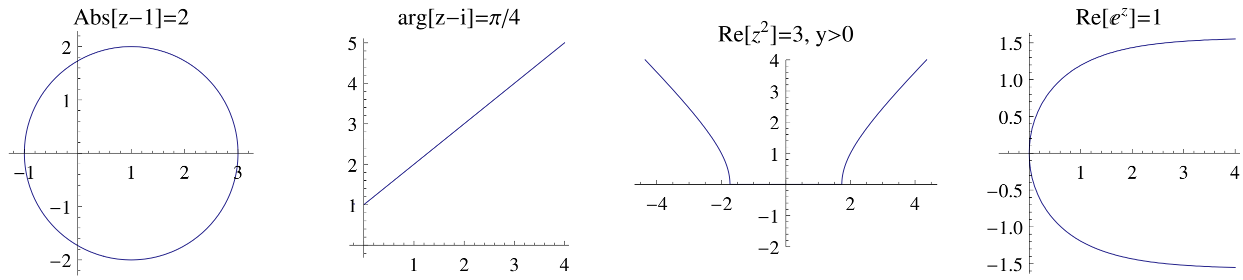
PHYS20672 Complex Variables and Vector Spaces: Solutions 1

1.
   a) $-(7/10) + (12/5)i$; $r = 5/2$,
      $\theta = 2\arctan(4/3) = 0.59\pi$
   b) $-(8/25) + (31/25)i$; $r = \sqrt{41}/5$,
      $\theta = \arctan(-31/8) = 0.58\pi$
   c) $3/2 - (3\sqrt{3}/2)i$; $r = 3$,
      $\theta = 5\pi/3 \equiv -\pi/3$
   d) $e/2 + (\sqrt{3}e/2)i$; $r = e$, $\theta = \pi/3$
   e) $\frac{1}{2}(3 - \sqrt{3}i)$; $r = \sqrt{3}$, $\theta = 11\pi/6 \equiv -\pi/6$



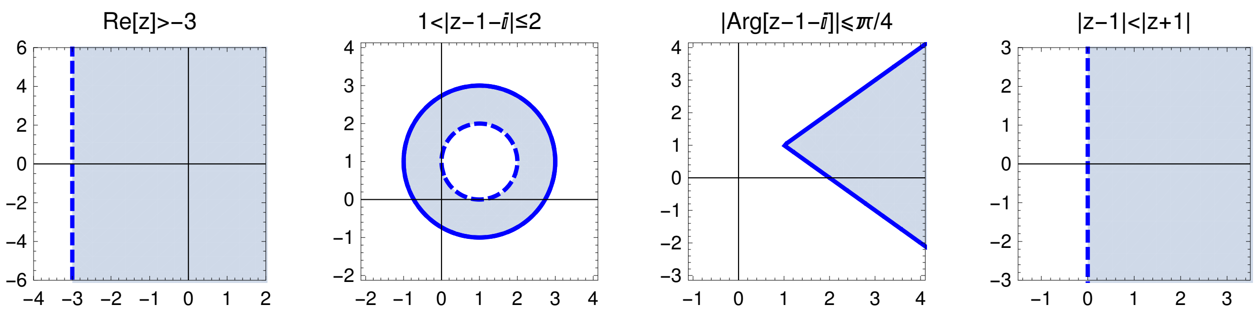
2. In terms of x and y, we have a) $(x-1)^2+y^2 = 4$; b) $y = x+1$ for $x > 0$; c) $x = \pm\sqrt{y^2+3}$, d) $x = -\ln(\cos y)$ for $-\pi/2 < y < \pi/2$. [Had (c) been for $x > 0$, only the RH branch would have appeared, symmetric under reflection in the x-axis.]



3. In (b) the boundaries are part of the region. For (d), we have

$$\sqrt{(x-1)^2 - y^2} < \sqrt{(x+1)^2 - y^2} \Rightarrow (x-1)^2 < (x+1)^2 \Rightarrow 0 < x.$$

So the region is $\operatorname{Re} z > 0$. More intuitively, we are looking for all points that are closer to $1 + 0i$ than to $-1 + 0i$, which is clearly all points to the right of the y-axis.



4. In the triangle inequality $a+b \geq c$, let $a=|z_1|$, $b=|z_2|$ and $c=|z_1 \pm z_2|$. The inequality $|z_1 \pm z_2| \leq |z_1|+|z_2|$ follows at once.

   We also have $a \leq b+c$ and $b \leq c+a$ for any triangle, so that $a-b \leq c$ and $b-a \leq c$. Of these last two inequalities, one will have a non-negative left-hand side equal to $|a-b|$. Thus, $|a-b| \leq c$, which gives $||z_1|-|z_2|| \leq |z_1 \pm z_2|$.

   a) In this case the two complex numbers are $z_1 = R^2 e^{i2\theta}$ and $z_2 = 1$, so $|z_1| = R^2$ and $|z_2| = 1$ respectively.

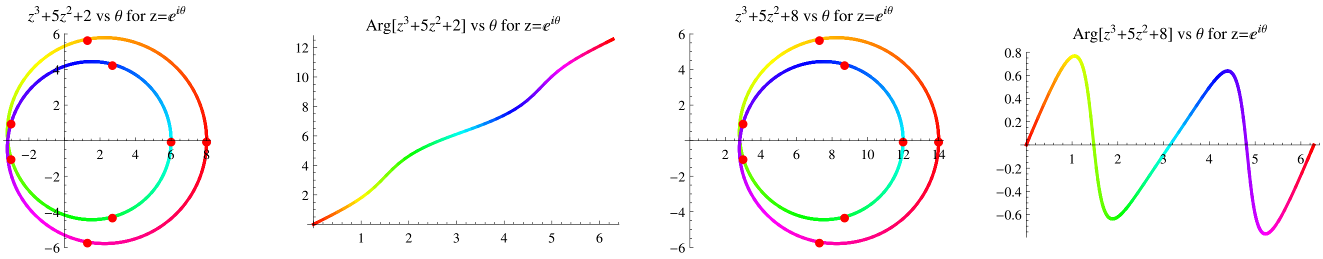
   b) Note that for any two complex numbers z_1 and z_2, $|z_1/z_2| = |z_1|/|z_2|$. The maximum value of the ratio will come from the maximum value of the numerator and the minimum value of the denominator.

5. a) $\pm 2^{1/4} e^{i\pi/8} = \pm(1.10 + 0.46i)$
   b) $\frac{1}{2}\ln 2 + i\pi/4 = 0.347 + 0.785i$
   c) $\cos(\pi/4 + i) = \cos(\pi/4)\cos i - \sin(\pi/4)\sin i = (1/\sqrt{2})(\cosh 1 - i\sinh 1) = 1.09 - 0.83i$
   d) If $z = \arcsin i$, then $i = \sin z = -i\sinh(iz)$. Hence $\sinh(iz) = -1$ and $z = i\,\mathrm{arcsinh}\,1 = 0.881i$ is a solution. There are many possible answers because the identities $\sin z = \sin(\pi - z)$ and $\sin z = \sin(z + 2\pi k)$ still hold for complex z.

6. a) $\sinh(iz) = \frac{1}{2}(e^{iz} - e^{-iz}) = i\sin z$
   b) $\sin(iz) = \frac{1}{2i}(e^{-z} - e^{z}) = i\sinh z$
   c) Let $iw = \arcsin(iz)$. Then $z = -i\sin(iw) = \sinh w$ and so $w = \mathrm{arcsinh}\,z$. Hence $\arcsin(iz) = i\,\mathrm{arcsinh}\,z$.
   d) Let $w = \mathrm{arcsinh}\,z$. Then $2z = e^w - e^{-w} \Rightarrow e^{2w} - 2ze^w - 1 = 0$. Hence $e^w = z \pm \sqrt{1+z^2}$ and $w = \mathrm{arcsinh}\,z = \ln\big(z + \sqrt{1+z^2}\big)$. (This is another example where there is more than one answer: however, taking the positive root and the principal value of the logarithm gives the branch for which arcsinh of a real number is real.)
   e) Let $w = \mathrm{arctanh}\,z$. Then $z = (e^w - e^{-w})/(e^w + e^{-w}) \Rightarrow e^{2w} = (1+z)/(1-z)$. Hence $\mathrm{arctanh}\,z = \frac{1}{2}\ln\big((1+z)/(1-z)\big)$.
   $\cos^2 z + \sin^2 z = \frac{1}{4}\big((e^{iz} + e^{-iz})^2 - (e^{iz} - e^{-iz})^2 = e^{iz}e^{-iz} = 1$ whether z is real or complex.

7. a) The domain of $1/(z^2+1)$ is the whole complex plane excluding $z = \pm i$
   b) The domain of $z/(z+\overline{z})$ is the complex plane excluding the imaginary axis $x = 0$ (where the denominator vanishes). c) The domain of $1/(|z|^2 - 1)$ is the complex plane excluding the points for which $|z| = 1$. d) The domain of $\mathrm{Ln}(z)$ is the complex plane excluding the origin, and with the restriction $-\pi < \theta \leq \pi$.) In (a) and (d) the domain is open and connected.

8. In the plots below, the lines change colour as θ increases to help trace the path. For $f(z) = z^3 + 5z^2 + 2$ the total phase change as θ increases from 0 to 2π is $\Delta\phi = 4\pi$ and so there must be two zeros inside the unit circle. (The roots are actually at -5.077 and $0.039 \pm 0.626i$). For $f(z) = z^3 + 5z^2 + 8$ the path in the w plane doesn't encircle the origin at all, so $\Delta\phi = 0$ and there are no zeros inside the unit circle. (The roots are at -5.286 and $0.143 \pm 1.222i$).



9. $w = z^2$: $u = x^2 - y^2$ and $v = 2xy$. So in the z plane, lines of constant $u = a$ are hyperbolae of the form $x = \pm\sqrt{a + y^2}$ and lines of constant $v = b$ are hyperbolae of the form $y = b/(2x)$.

   $w = \operatorname{Ln} z$: $u = \frac{1}{2}\ln(x^2 + y^2)$ and $v = \arctan(y/x)$. Then in the z plane, curves of constant $u = a$ are circles of radius e^a and lines of constant $v = b$ (where $-\pi < b \leq \pi$) are straight lines which make an angle $\tan b$ with the real axis.

   The two plots on the left show lines of constant u (blue) and v (red) in the complex z plane. The heavy lines are $u = 1$ and $v = 1$ and the black dot is the point $w = 0$.