PHYS20672 Complex Variables and Vector Spaces: Examples 7

Lower priority: ‡. Lowest priority: ‡‡.

65. The operator $\hat{K}$ is defined as follows by its action on functions of x:

$$\langle x|\hat{K}|g\rangle = -i\frac{\mathrm{d}g(x)}{\mathrm{d}x},$$

where $g(x) \equiv \langle x|g\rangle$. Show that $\hat{K}$ is Hermitian on the space of functions that vanish for $x \to \pm\infty$. Find the eigenvalues and eigenfunctions of $\hat{K}$, and show that these eigenfunctions are not square-integrable.

66. Show that $\hat{K}$, defined as above, is Hermitian on the space of functions $g(x)$ defined on the interval $[-\pi, \pi]$ which satisfy the periodic boundary condition $g(-\pi) = g(\pi)$. Find the eigenvalues and eigenfunctions of $\hat{K}$. Normalize the eigenfunctions to unity.

[In this question, unlike in Q.65, the eigenfunctions of $\hat{K}$ form a discrete set, and are normalizable. This illustrates the important part played by the choice of domain and boundary conditions.]

67. Kets $|f\rangle$ and $|g\rangle$ correspond to periodic functions defined as in Q.66 and $\{|e_n\rangle\}$ correspond to the normalized eigenfunctions of $\hat{K}$. If $|f\rangle = \sum_n f_n|e_n\rangle$, show that

$$\text{(i) } f_n = \langle e_n|f\rangle \qquad \text{(ii) } \langle f|g\rangle = \sum_n \overline{f_n}\, g_n \qquad \text{(iii) } \langle f|f\rangle = \sum_n |f_n|^2.$$

68. Show that $f(x)$ $(= \langle x|f\rangle)$ is the inverse Fourier transform of $\tilde{f}(k) = \langle e_k|f\rangle$, where, as in lectures, $\langle x|e_k\rangle = e^{ikx}/\sqrt{2\pi}$. (You can do this by inserting an appropriate resolution of unity between $\langle x|$ and $|f\rangle$.)

69. ‡ Let $u_n(x)$ be a normalized harmonic oscillator eigenfunction, i.e., the function of x that satisfies

$$-\frac{\mathrm{d}^2u_n}{\mathrm{d}x^2} + x^2u_n = (2n+1)u_n$$

and $\langle u_n|u_n\rangle = 1$. The solution for each n is unique, up to a constant factor of unit modulus. By transforming the differential equation to the k-representation, show that $\tilde{u}_n(k)$ satisfies

$$k^2\tilde{u}_n - \frac{\mathrm{d}^2\tilde{u}_n}{\mathrm{d}k^2} = (2n+1)\tilde{u}_n.$$

Use this result to argue that $\tilde{u}_n(s) = Cu_n(s)$, where $|C| = 1$. That is, the harmonic oscillator eigenfunctions and their Fourier transforms are the same, up to factors of unit modulus.

The next problem shows that it is surprisingly easy to construct other functions with the property of being equal to their own Fourier transform, up to a factor C, and that there are only four possible values for C.

70. An operator $\hat{F}$ is defined as follows by its action on square-integrable functions $g(x)$:

$$\langle x|\hat{F}|g\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixx'} g(x')\,\mathrm{d}x'.$$

[Yes, it's just Fourier transformation, but viewed here as an *active* transformation: $\hat{F}$ takes a function of x and converts it to a *new* function of x.]

(i) Show that the operator $\hat{F}$ is unitary. You can do this by verifying that $\hat{F}$ preserves the scalar product between arbitrary kets $|f\rangle$ and $|g\rangle$.

[This result is just Parseval's theorem for Fourier transforms, so you can consult your notes and change the notation slightly.]

(ii) Show that $\hat{F}^4 = \hat{1}$; i.e., show that if you Fourier-transform a function four times, you get back the original function.

(iii) Use the eigenvalue equation $\hat{F}|u\rangle = \omega|u\rangle$ and the result of part (ii) to prove that the eigenvalues of $\hat{F}$ are $\omega = 1$, i, -1 and $-i$.

[This illustrates a general principle: if $\hat{F}$ satisfies an algebraic operator equation, such as $\hat{F}^4 = \hat{1}$, its eigenvalues satisfy "the same" equation, in this case $\omega^4 = 1$. Compare this with the result stated in Q.64 on the last sheet.]

(iv) Prove that, for arbitrary $|g\rangle$, the vector $|u\rangle$ defined by

$$|u\rangle = \tfrac{1}{4}\big(\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big)|g\rangle$$

is an eigenvector of $\hat{F}$ with eigenvalue 1. Thus, the function $u(x) = \langle x|u\rangle$ is equal to its own Fourier transform.

(v) ‡ Show that the operator

$$\hat{P}_1 = \tfrac{1}{4}\big(\hat{1} + \hat{F} + \hat{F}^2 + \hat{F}^3\big)$$

is a *projection operator*, i.e., that it satisfies $\hat{P}_1^2 = \hat{P}_1$. It projects functions onto the set of functions that satisfy $\hat{F}|u\rangle = |u\rangle$.

(vi) ‡‡ Construct three more operators, $\hat{P}_\omega$, that project onto the sets of functions that satisfy $\hat{F}|u\rangle = \omega|u\rangle$, where $\omega = i$, -1 and $-i$. For the operators you have constructed, verify the completeness relation

$$\hat{P}_1 + \hat{P}_i + \hat{P}_{-1} + \hat{P}_{-i} = \hat{1}.$$