PHYS20672 Complex Variables and Vector Spaces: Examples 6

Lower priority: ‡. Lowest priority: ‡‡.

44. $\mathbb{C}^n$ is the set of all n-tuples of complex numbers, $(z_1, z_2, \ldots, z_n)$. If addition and multiplication by a scalar are defined in the usual way for a $1 \times n$ matrix (or row vector), show that $\mathbb{C}^n$ is a complex vector space.

45. The set of all real functions f, g, ...of a real variable $x \in [0, 1]$ is a vector space over $\mathbb{R}$: the sum of two functions, $h = f + g$, is defined so that h, when applied to x, gives $f(x) + g(x)$; the zero function, o, can be defined by $o(x) = 0$; and the additive inverse of a function f can be defined as the function that returns $-f(x)$ when applied to x. The product of f with a real scalar, $k = \lambda f$, is defined so that $k(x) = \lambda\big(f(x)\big)$. The required properties of commutativity, associativity and distributivity all follow from the corresponding properties for addition and multiplication of real numbers.

    If boundary conditions are specified, this can affect whether or not the functions form a vector space. Which of the following form a real vector space?

    (i) the set of real functions $f(x)$, where $x \in [0, 1]$ and $f(0) = f(1) = 0$.

    (ii) the set of real functions satisfying the periodic boundary condition $f(0) = f(1)$.

    (iii) the set of real functions for which $f(0) = 1$.

46. Are the following three vectors linearly independent: $|a\rangle = (2, 3, -1)$, $|b\rangle = (0, 1, 2)$, $|c\rangle = (0, 0, -5)$? [Write $\alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle = |0\rangle$ and prove that the only solution is $\alpha = \beta = \gamma = 0$.] Find the coordinates of $(2, -3, 1)$ relative to the basis $\{|a\rangle, |b\rangle, |c\rangle\}$.

47. Prove that the set of all polynomials in X of degree not exceeding 3, with complex coefficients, is a complex vector space. [You can regard 0 as an honorary polynomial of degree zero.] Write down the additive inverse of the vector $1 + iX + (2 + 3i)X^3$. Find a basis for the space and hence determine its dimension.
    What changes if only *odd* functions of X are considered?
    Why is the set of *cubic* polynomials <u>not</u> a vector space?

48. ‡‡ In this course, the scalars that appear in the definition of a vector space are generally assumed to be either real or complex numbers, since these are the cases are most often met in physical applications. But this need not always be the case. For example, consider the set $\mathbb{W}$ of real numbers of the form $x = p + q\sqrt{2}$, where p and q are *rational* numbers (i.e., p and q can be written as ratios of integers). Just as for real and complex numbers, addition, subtraction, multiplication and division can all be defined for rational numbers, with the exception (as usual) of division by zero.
    Show that $\mathbb{W}$ is a two-dimensional vector space, if the scalars are taken to be the set of rational numbers. In particular, explain why the zero vector $|0\rangle$ is unique.

49. For an arbitrary vector space,

    (i) show that the zero vector $|0\rangle$ is unique. [You need to show that if there is a second vector $|0'\rangle$ satisfying $|a\rangle + |0'\rangle = |a\rangle$, then $|0'\rangle = |0\rangle$.]

    (ii) show that $|-a\rangle$, the additive inverse of vector $|a\rangle$, is unique.

    (iii) show that $0|a\rangle = |0\rangle$ for any vector $|a\rangle$.

50. Regard the row vectors in Q.46 as ordinary Cartesian 3-vectors, so that for example $|a\rangle \equiv \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Apply the method of Gram–Schmidt orthogonalization to the vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ to obtain an orthonormal set. [In this problem the scalar product is the usual dot product, $\langle a|b\rangle \equiv \mathbf{a}\cdot\mathbf{b}$.]

51. (i) Prove the Schwarz inequality,

$$|\langle a|b\rangle| \leq |a||b|.$$

    [If either vector is zero, there is nothing to prove (why not?), so you can assume that $|b\rangle$ is nonzero. Define $|c\rangle = |a\rangle - \lambda|b\rangle$ with $\lambda = \langle b|a\rangle/\langle b|b\rangle$. Then use $\langle c|c\rangle \geq 0$.]

    (ii) Use the Schwarz inequality to prove the triangle inequality,

$$|c| \leq |a| + |b| \quad \text{if} \quad |c\rangle = |a\rangle + |b\rangle.$$

    [Both sides are nonnegative, so you can consider the squares of each side.]
    Use the triangle inequality to show that if $|c\rangle$ is defined as above, then

$$||a| - |b|| \leq |c|.$$

    [Note that "$|\cdots|$" has two different meanings in the line above.]

52. Let $|a\rangle = 3i|e_1\rangle - 7i|e_2\rangle$ and $|b\rangle = |e_1\rangle + 2|e_2\rangle$, where $|e_1\rangle$ and $|e_2\rangle$ are orthonormal. Show explicitly that $|a\rangle$ and $|b\rangle$ satisfy the triangle and Schwarz inequalities.

53. Let $\{|e_i\rangle\}$ be an orthonormal basis of $\mathbb{V}^N$, so that we can write $|a\rangle = \sum_i a_i|e_i\rangle$ and $|b\rangle = \sum_i b_i|e_i\rangle$. Show that

    (i) $b_i = \langle e_i|b\rangle$ and $\overline{a_i} = \langle a|e_i\rangle$.

    (ii) $\langle a|b\rangle = \sum_i \overline{a_i} b_i = \overline{\langle b|a\rangle}$.

54. The scalar product of two real functions $u(x)$ and $v(x)$, where $x \in [-1, 1]$, is defined by

$$\langle u|v\rangle = \int_{-1}^{1} u(x)v(x)\,\mathrm{d}x.$$

    Let $p_n(x)$ be polynomials of degree n in x. Given that these polynomials form an orthonormal set of functions on the interval $[-1, 1]$, find p_0, p_1 and p_2. [The results are unique up to the choice of sign.] Where have you seen these polynomials before, perhaps with a different normalization?

55. Show that if $|a\rangle$ is a nonzero vector in $\mathbb{V}^N$, the set $\mathbb{W}$ of vectors orthogonal to $|a\rangle$ is a vector space of dimension $N - 1$. [Since every vector in $\mathbb{W}$ is also a vector in $\mathbb{V}^N$, we say that $\mathbb{W}$ is a **subspace** of $\mathbb{V}^N$.]

56. ‡ Let $\{|a_i\rangle\}$ be a set of N vectors spanning $\mathbb{V}^N$. This basis is **not** necessarily orthogonal.

   (i) Prove that for each $|a_i\rangle$, a so-called *reciprocal vector* $|\breve{a}_i\rangle$ can be constructed such that $\langle\breve{a}_i|a_j\rangle = \delta_{ij}$. [You could use the Gram–Schmidt procedure to construct an orthonormal basis for the subspace spanned by $\{|a_j\rangle,\ j \neq i\}$, then construct $|\breve{a}_i\rangle$ from these unit vectors and $|a_i\rangle$. Other methods are possible.]

   (ii) Show that the formula
$$\sum_i |a_i\rangle\langle\breve{a}_i| = \hat{1}$$
   generalizes the completeness relation $\sum_i |e_i\rangle\langle e_i| = \hat{1}$ to a non-orthonormal basis.

   (iii) Use the result of (ii) to show that if $\hat{A}|a_i\rangle = |b_i\rangle$, then
$$\hat{A} = \sum_i |b_i\rangle\langle\breve{a}_i| = \sum_{i,j} |a_i\rangle\langle\breve{a}_i|\hat{A}|a_j\rangle\langle\breve{a}_j|.$$

   (iv) Show that if the vectors $\{|b_i\rangle\}$ are linearly independent, then the inverse operator can be constructed as
$$\hat{A}^{-1} = \sum_i |a_i\rangle\langle\breve{b}_i|.$$
   What goes wrong if they are **not** linearly independent?

   [Reciprocal vectors are used frequently in solid state physics, because crystal axes are typically not orthogonal (most crystals are not cubic). But simply knowing that they can be constructed is sometimes useful elsewhere – Q.61 gives an example.]

57. Show that the set of linear operators on a vector space $\mathbb{V}$ is itself a vector space.

58. ‡ Let $\hat{U}$ be a unitary operator on $\mathbb{V}^N$. If its eigenvalues ω_i are all different from one another, show that the eigenvectors of $\hat{U}$ are orthogonal. Also show that the eigenvalues have unit modulus, $|\omega_i| = 1$.

59. For each of the following matrices, state whether it is Hermitian, unitary, both, or neither:
$$\text{(i)}\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\quad \text{(ii)}\ \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix};\quad \text{(iii)}\ \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix};\quad \text{(iv)}\ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},\ b \neq 0.$$
   Calculate the eigenvalues and eigenvectors in each case, verifying orthogonality of the eigenvectors where appropriate. Comment on your results for (iv).

60. An example of a 3D rotation matrix is
$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
   Find its eigenvalues and normalized eigenvectors and verify that they have the properties expected for a unitary matrix. [These properties are stated in Q.58.]

61. ‡ *This question should be of interest if you tried Q.56 on reciprocal vectors.*

Consider the eigenvalue problem

$$\hat{A}|u_i\rangle = \lambda_i|u_i\rangle,$$

where $\hat{A}$ is an operator on $\mathbb{V}^N$ and the set of eigenvectors $\{|u_i\rangle\}$ spans $\mathbb{V}^N$. Such an operator is said to be **diagonalizable**; see part (c). In general, the eigenvectors of a diagonalizable operator need not be orthogonal.

(a) Use the eigenvalue equation to show that

$$\hat{A} = \sum_i \lambda_i\,|u_i\rangle\langle\breve{u}_i|,$$

where $|\breve{u}_i\rangle$ is the vector reciprocal to $|u_i\rangle$; see Q.56. This generalizes the spectral representation to operators that are neither Hermitian nor unitary.

(b) Use the spectral representation of $\hat{A}$ to show that the bra vectors $\{\langle\breve{u}_i|\}$ satisfy the equation

$$\langle\breve{u}_i|\hat{A} = \lambda_i\langle\breve{u}_i|\,;$$

we say that $\langle\breve{u}_i|$ is a *left* eigenvector of $\hat{A}$ and that $|u_i\rangle$ is a *right* eigenvector.

(c) ‡‡ Show that the matrix $\mathbf{A}$ with elements $A_{ij} = \langle e_i|\hat{A}|e_j\rangle$, where $\{|e_i\rangle\}$ is a complete, orthonormal set of vectors, can be diagonalized by the transformation

$$\mathbf{A}^{\text{diag}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S},$$

where $\mathbf{S}$ is a matrix whose columns are the right eigenvectors of $\mathbf{A}$. What can you say about the rows of $\mathbf{S}^{-1}$? [A transformation of this kind is called a **similarity transformation**; a unitary transformation is a particular kind of similarity transformation.]

62. Show that for any unitary matrix $\mathbf{U}$, $|\det\mathbf{U}| = 1$.

[Note that for any square matrix $\mathbf{A}$, $\det(\mathbf{A}^{\mathsf{T}}) = \det\mathbf{A}$. Use this to show that $\det(\mathbf{U}^{\dagger}) = \overline{\det\mathbf{U}}$. Then use $\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{I}$ to prove that $|\det\mathbf{U}|^2 = 1$.]

63. ‡ For a matrix $\mathbf{M}$ that is either Hermitian or unitary, use the fact that $\mathbf{M}$ can be diagonalized by a unitary transformation to show that the trace of $\mathbf{M}$ equals the sum of its eigenvalues and its determinant equals the product of its eigenvalues.
Use these results to show that

$$\det[\exp(\mathbf{M})] = \exp[\mathrm{Tr}(\mathbf{M})]$$

for the cases where $\mathbf{M}$ is Hermitian or unitary.

‡‡ If you have tried Q.61(c), generalize to the case of a diagonalizable matrix. [The result also applies to square matrices that are *not* diagonalizable.]

64. ‡ The characteristic equation satisfied by the eigenvalues of an $N \times N$ matrix $\mathbf{M}$ is the polynomial equation

$$\det(\lambda \mathbf{I} - \mathbf{M}) = \sum_{r=0}^{N} c_r \lambda^r = 0.$$

The Cayley–Hamilton theorem states that $\mathbf{M}$ satisfies an equation of the same form, i.e.

$$\sum_{r=0}^{N} c_r \mathbf{M}^r = \mathbf{0},$$

in which $\mathbf{M}^0$ is interpreted as the unit matrix $\mathbf{I}$.

Prove the Cayley–Hamilton theorem for the cases where $\mathbf{M}$ is Hermitian or unitary (or, more generally, diagonalizable ‡‡). It is a potentially useful result, because it means that $\mathbf{M}^N$ (and any higher power of $\mathbf{M}$) can be expressed as a linear combination of the matrices $\mathbf{M}^0$ to $\mathbf{M}^{N-1}$. The theorem also applies to square matrices $\mathbf{M}$ that cannot be diagonalized, but this further generalization is not very easy to prove.