PHYS20672 Complex Variables and Vector Spaces: Examples 4

Lower priority: ‡. Lowest priority: ‡‡. Harder problem, but still good practice: *.

29. By using the formulas for the coefficients a_n and b_n in terms of contour integrals, show that the Laurent series of $1/(z+2)$ about $z=1$, for $|z-1|>3$, is

$$\frac{1}{z+2}=\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{(z-1)^n}=\frac{1}{z-1}-\frac{3}{(z-1)^2}+\frac{9}{(z-1)^3}-\dots$$

Note that this was derived in lectures using a geometric series.

Hints: The contour of integration C is $|z-1|=R$, for any $R>3$. For a_n the integrand is $\dfrac{1}{(z+2)(z-1)^{n+1}}$ which has a simple pole at $z=-2$ and a pole of order $n+1$ at $z=1$, both within the contour. For b_n the integrand is $\dfrac{(z-1)^{n-1}}{z+2}$ which only has a simple pole at $z=-2$. In each case use the appropriate Cauchy integral formulae to evaluate the integrals, splitting the contour into smaller ones circling each pole as required.

The rest of the questions on this sheet do not require the use of the explicit contour-integral definitions of the coefficients a_n and b_n.

30. Without any calculation, explain why the Taylor expansion of $\tan z$ about $z=\pi/4$ must have a radius of convergence equal to $\pi/4$.

‡‡ Find the first four terms of the Taylor expansion of $\tan z$ about $z=\pi/4$. (Note that $\tan z$ is analytic near $z=\pi/4$, so you can use the usual expression for the coefficients in terms of derivatives of the function.)

31. If z_0 is a non-zero complex number, find the Taylor series of $f(z)=1/(z-z_0)$ about $z=0$, and show that its radius of convergence is $|z_0|$. (Note that $f(z)$ is analytic in the vicinity of $z=0$.) Check that the result agrees with that obtained using a geometric series. Use a geometric series to find the Laurent series in the region $|z|>|z_0|$.

32. Use the results above, and partial fractions, to find the Taylor or Laurent series, as appropriate, of $f(z)=\dfrac{z+1}{(z-2)(z-3)}$

(a) about $z=0$, for the region $|z|<2$
(b) about $z=0$, for the region $2<|z|<3$
(c) about $z=0$, for the region $3<|z|$
(d) about $z=2$, for the region $1<|z-2|$ (Hint: change variable to $w=z-2$.)
(e) about $z=1$, for the region $|z-1|<1$
(f) about $z=1$, for the region $1<|z-1|<2$
(g) about $z=1$, for the region $2<|z-1|$.

Hint: in (b) we need a Taylor series for the $(z-3)^{-1}$ term, but a Laurent series for the $(z-2)^{-1}$ term. The same idea is needed in some later parts as well.

‡ With the help of a computer (‡‡ or otherwise), explore numerically the convergence or lack of it of the series (a)–(c), for $z=1$, $z=2.5$ and $z=6$.

33. Find the Laurent series about $z = 0$ of the following functions:

$$\text{(a) } \sin(1/z) \qquad \text{(b) } \ddagger\ z^{-3}\sin^2 z \qquad \text{(c) } z^3 e^{1/z} \qquad \text{(d) } z^{-2}(\cos z - 1)$$

In each case, describe the nature of the singularity. Also, give the value of the coefficient of $1/z$; i.e., b_1, also known as the *residue*. To avoid explicit contour integration, you should use standard Taylor series for functions such as e^z where appropriate.

34. Evaluate the residues of $f(z) = \dfrac{z^2+1}{z(z-1)^3}$ at each of its singularities.

35. Evaluate the residues at each of the singularities of the following functions:

$$\text{(a) } \frac{z^2+z-2}{(z-1)^2} \qquad \text{(b) } \frac{1}{\cos z} \qquad \text{(c) } \ddagger\ \frac{z}{\sin^2 z} \qquad \text{(d) } \ddagger\ \frac{1}{\sin z - \cos z} \qquad \text{(e) } \frac{1}{\sinh z}$$

For part (e), note that $\sinh(a+b) = \sinh a\,\cosh b + \cosh a\,\sinh b$.