PHYS20672 Complex Variables and Vector Spaces: Examples 1

These are mostly problems for basic practice, so there is no prioritization.

1. Write the following expressions in the form $x+iy$ and sketch their locations on the complex plane:

$$\text{(a) } \tfrac{1}{10}(3+4i)^2 \qquad \text{(b) } \frac{4+5i}{3-4i} \qquad \text{(c) } 3e^{5i\pi/3} \qquad \text{(d) } e^{1+i\pi/3} \qquad \text{(e) } 1+ie^{7i\pi/6}$$

   For each case find $|z|$ and the value of the argument θ (for $0 \le \theta < 2\pi$).

2. Sketch the curves

$$\text{(a) } |z-1| = 2 \qquad \text{(b) } \arg(z-i) = \pi/4$$
$$\text{(c) } \mathrm{Re}(z^2) = 3 \text{ for } y > 0 \qquad \text{(d) } \mathrm{Re}(e^z) = 1 \text{ for } -\pi/2 < y < \pi/2.$$

3. The inequalities given below specify sets of points in the complex plane. Make sketches to illustrate these sets and indicate which boundaries (if any) form part of each set.

$$\text{(a) } \mathrm{Re}\, z > -3 \qquad \text{(b) } 1 < |z-1-i| \le 2 \qquad \text{(c) } |\arg(z-1-i)| \le \pi/4 \qquad \text{(d) } |z-1| < |z+1|.$$

4. Give a geometric argument to show that for any two complex numbers z_1 and z_2,

$$|(|z_1| - |z_2|)| \le |z_1 \pm z_2| \le |z_1| + |z_2|.$$

   [Hint: draw a triangle with sides z_1, z_2 and $z_1 \pm z_2$. Make use of the inequality $a + b \ge c$ for the three sides of a triangle.]

   Hence show that on the circle $|z| = R$, with $R > 1$,

$$\text{(a) } R^2 - 1 \le |z^2 \pm 1| \le R^2 + 1 \qquad \text{(b) } \left|\frac{z^2+1}{z^2-1}\right| \le \frac{R^2+1}{R^2-1}.$$

5. Calculate the following:

$$\text{(a) } \sqrt{1+i} \qquad \text{(b) } \mathrm{Ln}(1+i) \qquad \text{(c) } \cos(\pi/4+i) \qquad \text{(d) } \arcsin i$$

6. Verify the following identities:

$$\text{(a) } \sinh(iz) = i\sin z \qquad \text{(b) } \sin(iz) = i\sinh z \qquad \text{(c) } \arcsin(iz) = i\,\mathrm{arcsinh}\, z$$
$$\text{(d) } \mathrm{arcsinh}\, z = \ln\big(z + \sqrt{1+z^2}\big) \qquad \text{(e) } \mathrm{arctanh}\, z = \tfrac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

   Show that $\cos^2 z + \sin^2 z = 1$ even for complex z.

7. For each of the following functions, specify the domain, i.e. the set of complex numbers for which the function can be defined and is finite:

$$\text{(a) } f(z) = \frac{1}{z^2+1} \qquad \text{(b) } f(z) = \frac{z}{z+\overline{z}} \qquad \text{(c) } \frac{1}{|z|^2-1} \qquad \text{(d) } \mathrm{Ln}\, z$$

   For which is the domain an open, connected set of points of the complex plane?

8. Consider the function $f(z) = z^3 + 5z^2 + 2$. Calculate (numerically, e.g. using Mathematica) $f(z)$ for $z = \exp(in\pi/4)$ for $n = 0$ to 8. Hence sketch the path traced in the w-plane for $w = f(z)$ as z follows the unit circle, with $0 \le \theta < 2\pi$. Also, sketch a plot of $\arg w$ as a function of θ. Repeat for the function $z^3 + 5z^2 + 8$. Relate your results for the increase in $\arg w$ to the number of zeros of each function with a modulus less than one.

9. For the functions $w = z^2$ and $w = \mathrm{Ln}\, z$, plot lines of constant u and v in the complex plane $z = x + iy$. If you want to compare with Handout 1, note that in the handout lines of constant x and y were plotted in the w plane; thus, you should compare with the plots of the inverse functions.